Confluence for Local Lambda-Mu Calculus

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Abstract

This note is taken directly from my comprehensive exam, modulo some re-organizations.

1 Hardin's Interpretation Method

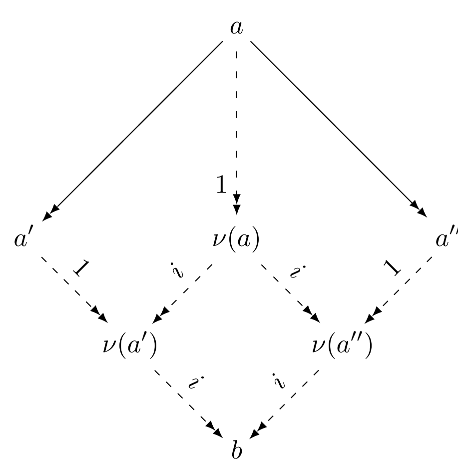
Sometimes it is inevitable to deal with reduction systems that contains more than one reduction, for example, $(\Lambda, \{\to_\beta, \to_\eta\})$. Confluence problem for this kind of system require some nontrivial efforts to prove. Hardin's interpretion method [3] provide a way to deal with some of those reduction systems.

Lemma 1 (Interpretation lemma)**.** *Let $\to$ be $\to_1 \cup \to_2$, $\to_1$ being confluent and strongly normalizing. We denote by $\nu(a)$ the $\to_1$-normal form of a. Suppose that there is some relation $\to_i$ on $\to_1$ normal forms satisfying:*

$\to_i \subseteq \twoheadrightarrow$, *and* $a \to_2 b$ *implies* $\nu(a) \twoheadrightarrow_i \nu(b)$ (†)

Then the confluence of $\to_i$ implies the confluence of $\to$.

Proof. So suppose $\to_i$ is confluent. If $a \twoheadrightarrow a'$ and $a \twoheadrightarrow a''$. So by (†), $\nu(a) \twoheadrightarrow_i \nu(a')$ and $\nu(a) \twoheadrightarrow_i \nu(a'')$. Notice that $t \to_1^* t'$ implies $\nu(t) = \nu(t')$(By confluence and strong normalizing of $\to_1$). By confluence of $\to_i$, there exists b such that $\nu(a') \twoheadrightarrow_i b$ and $\nu(a'') \twoheadrightarrow_i b$. Since $\to_i, \to_1 \subseteq \twoheadrightarrow$, we got $a' \twoheadrightarrow \nu(a') \twoheadrightarrow b$ and $a'' \twoheadrightarrow \nu(a'') \twoheadrightarrow b$. Hence $\to$ is confluent.



□

Hardin's method reduce the confluence problem of $\to_1 \cup \to_2$ to $\to_i$, given the confluence and strong normalizing of $\to_1$, this make it possible to apply Tait-Martin-Löf's (Takahashi's) method to prove confluence of $\to_i$.

2 Local $\lambda\mu$ Calculus

We now show an applicaiton of Hardin's method on a concrete example, this example arise naturally in proving type preservation for Selfstar. The approach we adopt is similar to the one in [2].

Definition 1 (Local Lambda Mu Terms)**.**
Terms $t \ ::= \ x \mid \lambda x.t \mid tt' \mid \mu t$
Closure $\mu \ ::= \{x_i \mapsto t_i\}_{i \in \mathcal{I}}$

The closure is basically a set of recursively defined definitions. Let $\mathcal{I}$ be a finite nonempty index set. For $\{x_i \mapsto t_i\}_{i\in\mathcal{I}}$, we require for any $1 \leq i \leq n$, the set of free variables of t_i, $\mathsf{FV}(t_i) \subseteq dom(\mu) = \{x_1, ..., x_n\}$ and we do not allow reduction, definition substitution, substitution inside the closure, we call it *local property*, without this property, we are in the dangerous of losing confluence property(see [1] for a detailed discussion). $\mu \in t$ means the closure μ appears in t. $\vec{\mu}t$ denotes $\mu_1 ... \mu_n t$. $[t'/x](\mu t) \ \equiv \mu([t'/x]t)$. So $\mathsf{FV}(\mu t) = \mathsf{FV}(t) - dom(\mu)$.

Definition 2 (Beta-Reductions)**.**

$$\frac{}{(\lambda x.t)t' \to_\beta [t'/x]t} \quad \frac{(x_i \mapsto t_i) \in \mu}{\mu x_i \to_\beta \mu t_i} \quad \frac{t \to_\beta t'}{\lambda x.t \to_\beta \lambda x.t'} \quad \frac{t \to_\beta t''}{tt' \to_\beta t''t'} \quad \frac{t' \to_\beta t''}{tt' \to_\beta tt''} \quad \frac{t \to_\beta t'}{\mu t \to_\beta \mu t'}$$

Definition 3 (Mu-Reductions)**.**

$$\frac{dom(\mu)\#\mathsf{FV}(t)}{\mu t \to_\mu t} \quad \frac{}{\mu(\lambda x.t) \to_\mu \lambda x.\mu t} \quad \frac{}{\mu(t_1 t_2) \to_\mu (\mu t_1)(\mu t_2)} \quad \frac{t \to_\mu t'}{\lambda x.t \to_\mu \lambda x.t'}$$

$$\frac{t' \to_\mu t''}{tt' \to_\mu tt''} \quad \frac{t \to_\mu t''}{tt' \to_\mu t''t'} \quad \frac{t \to_\mu t'}{\mu t \to_\mu \mu t'}$$

2.0.1 Confluence of Local λ_μ Calculus

Lemma 2. *$\to_\mu$ is strongly normalizing and confluent.*

Definition 4 (μ-Normal Forms)**.**
$n \ ::= \ x \mid \mu x_i \mid \lambda x.n \mid nn'$

We require $x_i \in dom(\mu)$.

Definition 5 (μ-Normalize Funciton)**.**

$m(x) \ := \ x$ $\qquad m(\lambda y.t) \ := \ \lambda y.m(t)$
$m(t_1 t_2) \ := \ m(t_1)m(t_2)$ $\qquad m(\vec{\mu}y) \ := y$ *if* $y \notin dom(\vec{\mu})$.
$m(\vec{\mu}y) \ := \mu_i y$ *if* $y \in dom(\mu_i)$. $\qquad m(\vec{\mu}(tt')) \ := m(\vec{\mu}t)m(\vec{\mu}t')$
$m(\vec{\mu}(\lambda x.t)) \ := \lambda x.m(\vec{\mu}t)$.

Lemma 3. *Let Φ denote the set of μ normal form, for any term t, $m(t) \in \Phi$.*

Proof. One way to prove this is first identify t as $\dot{\vec{\mu_1}}t'$, here $\dot{\vec{\mu_1}}$ means there are zero or more closures and t' does not contains any closure at head position. Then we can proceed by induction on the structure of t':

Base Cases: $t' = x$, obvious.

Step Cases: If $t' = \lambda x.t''$, then $m(\dot{\vec{\mu_1}}(\lambda x.t'')) \equiv \lambda x.m(\dot{\vec{\mu_1}}t'')$. Now we can again identify t'' as $\dot{\vec{\mu_2}}t'''$, where t''' does not have any closure at head position. Since t''' is structurally smaller than $\lambda x.t''$, by IH, $m(\dot{\vec{\mu_1}}\dot{\vec{\mu_2}}t''') \in \Phi$, thus $m(\dot{\vec{\mu_1}}(\lambda x.t'')) \equiv \lambda x.m(\dot{\vec{\mu_1}}t'') \in \Phi$.

For $t' = t_1 t_2$, we can argue similarly as above.

□

Definition 6 (β Reduction on μ-normal Forms)**.**

$$\frac{n \to_\beta t}{n \to_{\beta\mu} m(t)} \qquad \frac{n \to_{\beta\mu} n'}{\lambda x.n \to_{\beta\mu} \lambda x.n'} \qquad \frac{n' \to_{\beta\mu} n''}{nn' \to_{\beta\mu} nn''} \qquad \frac{n \to_{\beta\mu} n''}{nn' \to_{\beta\mu} n''n'}$$

Note that the last three rules follows from the first rule. For the second one, because $n \to_\beta t$ implies $\lambda x.n \to_\beta \lambda x.t$ and $m(\lambda x.t) \equiv \lambda x.m(t)$. The others follow similarly.

Definition 7 (Parallelization)**.**

$$\frac{}{n \Rightarrow_{\beta\mu} n} \qquad \frac{(x_i \mapsto t_i) \in \mu}{\mu x_i \Rightarrow_{\beta\mu} m(\mu t_i)} \qquad \frac{n_1 \Rightarrow_{\beta\mu} n_1' \quad n_2 \Rightarrow_{\beta\mu} n_2'}{(\lambda x.n_1)n_2 \Rightarrow_{\beta\mu} m([n_1'/x]n_2')}$$

$$\frac{n \Rightarrow_{\beta\mu} n'}{\lambda x.n \Rightarrow_{\beta\mu} \lambda x.n'} \qquad \frac{n' \Rightarrow_{\beta\mu} n''' \quad n \Rightarrow_{\beta\mu} n''}{nn' \Rightarrow_{\beta\mu} n''n'''}$$

Lemma 4. $\to_{\beta\mu} \subseteq \Rightarrow_{\beta\mu} \subseteq \to_{\beta\mu}^*$.

Lemma 5. *If $n_2 \Rightarrow_{\beta\mu} n_2'$, then $m([n_2/x]n_1) \Rightarrow_{\beta\mu} m([n_2'/x]n_1)$.*

Proof. By induction on the structure of n_1. We list a few non-trivial cases:

Base Cases: $n_1 = x$, $n_1 = \mu x_i$, Obvious.

Step Case: $n_1 = \lambda y.n$. We have $m(\lambda y.[n_2/x]n) \equiv \lambda y.m([n_2/x]n) \overset{IH}{\Rightarrow}_{\beta\mu} \lambda y.m([n_2'/x]n) \equiv m(\lambda y.[n_2'/x]n)$.

Step Case: $n_1 = nn'$. We have $m([n_2/x]n[n_2/x]n') \equiv m([n_2/x]n)m([n_2/x]n') \overset{IH}{\Rightarrow}_{\beta\mu} m([n_2'/x]n)m([n_2'/x]n') \equiv m([n_2'/x]n[n_2'/x]n)$.

□

Lemma 6. *$m(m(t)) \equiv m(t)$ and $m([m(t_1)/y]m(t_2)) \equiv m([t_1/y]t_2)$.*

Proof. The first equality is by lemma 3. For the second equality, we prove it through similar method as lemma 3: We identify t_2 as $\dot{\overrightarrow{\mu_1}}t_2'$, t_2' does not contains any closure at head position. We proceed by induction on the structure of t_2':

Base Cases: For $t_2' = x$, we use $m(m(t)) \equiv m(t)$.

Step Cases: If $t_2' = \lambda x.t_2''$, then $m(\dot{\overrightarrow{\mu_1}}(\lambda x.[t_1/y]t_2'')) \equiv \lambda x.m(\dot{\overrightarrow{\mu_1}}([t_1/y]t_2'')) \equiv \lambda x.m(\dot{\overrightarrow{\mu_1}}\dot{\overrightarrow{\mu_2}}([t_1/y]t_2''')),$ where t_2'' as $\dot{\overrightarrow{\mu_2}}t_2'''$ and t_2''' does not have any closure at head position. Since t_2''' is structurally smaller than $\lambda x.t_2''$, by IH, $m(\overrightarrow{\mu_1}\overrightarrow{\mu_2}([t_1/y]t_2''')) \equiv m([t_1/y](\dot{\overrightarrow{\mu_1}}\dot{\overrightarrow{\mu_2}}t_2''')) \equiv m([m(t_1)/y]m(\dot{\overrightarrow{\mu_1}}\dot{\overrightarrow{\mu_2}}t_2'''))$. Thus $\lambda x.m(\dot{\overrightarrow{\mu_1}}\dot{\overrightarrow{\mu_2}}([t_1/y]t_2''')) \equiv \lambda x.m([m(t_1)/y]m(\overrightarrow{\mu_1}\overrightarrow{\mu_2}t_2'''))$. So $m([t_1/y]\dot{\overrightarrow{\mu_1}}(\lambda x.t_2'')) \equiv m([m(t_1)/y]m(\lambda x.\dot{\overrightarrow{\mu_1}}\dot{\overrightarrow{\mu_2}}t_2''')) \equiv m([m(t_1)/y]m(\lambda x.\dot{\overrightarrow{\mu_1}}t_2'')) \equiv m([m(t_1)/y]m(\dot{\overrightarrow{\mu_1}}(\lambda x.t_2'')))$

For $t_2' = t_a t_b$, we can argue similarly as above.

□

Lemma 7. *If $n_1 \Rightarrow_{\beta\mu} n_1'$ and $n_2 \Rightarrow_{\beta\mu} n_2'$, then $m([n_2/x]n_1) \Rightarrow_{\beta\mu} m([n_2'/x]n_1')$.*

Proof. We prove this by induction on the derivation of $n_1 \Rightarrow_{\beta\mu} n_1'$.

Base Case:

$$\frac{}{n \Rightarrow_{\beta\mu} n}$$

By the lemma 5.

Base Case:

$$\frac{x_i \mapsto t_i \in \mu}{\mu x_i \Rightarrow_{\beta\mu} m(\mu t_i)}$$

Because $y \notin \mathsf{FV}(\mu x_i)$ and μ is local.

Step Case:

$$\frac{n_a \Rightarrow_{\beta\mu} n'_a \quad n_b \Rightarrow_{\beta\mu} n'_b}{(\lambda x.n_a)n_b \Rightarrow_{\beta\mu} m([n'_a/x]n'_b)}$$

We have $m((\lambda x.[n_2/y]n_a)[n_2/y]n_b) \equiv (\lambda x.m([n_2/y]n_a))m([n_2/y]n_b)$
$\stackrel{IH}{\Rightarrow}_{\beta\mu} m([m([n'_2/y]n'_b)/x]m([n'_2/y]n'_a)) \equiv m([n'_2/y]([n'_b/x]n'_a))$. The last equality is by lemma 6.

Step Case:

$$\frac{n \Rightarrow_{\beta\mu} n'}{\lambda x.n \Rightarrow_{\beta\mu} \lambda x.n'}$$

We have $m(\lambda x.[n_2/y]n) \equiv \lambda x.m([n_2/y]n) \stackrel{IH}{\Rightarrow}_{\beta\mu} \lambda x.m([n'_2/y]n') \equiv m(\lambda x.[n'_2/y]n')$

Step Case:

$$\frac{n_a \Rightarrow_{\beta\mu} n'_a \quad n_b \Rightarrow_{\beta\mu} n'_b}{n_a n_b \Rightarrow_{\beta\mu} n'_a n'_b}$$

We have $m([n_2/y]n_a[n_2/y]n_b) \equiv m([n_2/y]n_a)m([n_2/y]n_b)$
$\stackrel{IH}{\Rightarrow}_{\beta\mu} m([n'_2/y]n'_a)m([n'_2/y]n'_b) \equiv m([n'_2/y](n'_a n'_b))$.

□

Lemma 8. *If $n \Rightarrow_{\beta\mu} n'$ and $n \Rightarrow_{\beta\mu} n''$, then there exist n''' such that $n'' \Rightarrow_{\beta\mu} n'''$ and $n' \Rightarrow_{\beta\mu} n'''$. So $\to_{\beta\mu}$ is confluent.*

Proof. By induction on the derivation of $n \Rightarrow_{\beta\mu} n'$.
Base Case:

$$\frac{}{n \Rightarrow_{\beta\mu} n}$$

Obvious.

Base Case:

$$\frac{}{\mu x_i \Rightarrow_{\beta\mu} m(\mu t_i)}$$

Obvious.

Step Case:

$$\frac{n_1 \Rightarrow_{\beta\mu} n'_1 \quad n_2 \Rightarrow_{\beta\mu} n'_2}{(\lambda x.n_1)n_2 \Rightarrow_{\beta\mu} m([n'_1/x]n'_2)}$$

Suppose $(\lambda x.n_1)n_2 \Rightarrow_{\beta\mu} (\lambda x.n_1'')n_2''$, where $n_1 \Rightarrow_{\beta\mu} n_1''$ and $n_2 \Rightarrow_{\beta\mu} n_2''$. By lemma 7 and IH, we have $m([n_1'/x]n_2') \Rightarrow_{\beta\mu} m([n_1'''/x]n_2''')$. We also have $(\lambda x.n_1'')n_2'' \Rightarrow_{\beta\mu} m([n_1'''/x]n_2''')$, where $n_1'' \Rightarrow_{\beta\mu} n_1'''$ and $n_1' \Rightarrow_{\beta\mu} n_1'''$ and $n_2' \Rightarrow_{\beta\mu} n_2'''$ and $n_2' \Rightarrow_{\beta\mu} n_2'''$.

Suppose $(\lambda x.n_1)n_2 \Rightarrow_{\beta\mu} m([n_2''/x]n_1'')$, where $n_1 \Rightarrow_{\beta\mu} n_1''$ and $n_2 \Rightarrow_{\beta\mu} n_2''$. By lemma 7 and IH, we have $m([n_1'/x]n_2') \Rightarrow_{\beta\mu} m([n_1'''/x]n_2''')$ and $m([n_1''/x]n_2'') \Rightarrow_{\beta\mu} m([n_1'''/x]n_2''')$.

The other cases are either similar to the one above or easy.

□

One can also use Takahashi's method([4]) to prove the lemma above. We will not explore that here.

Lemma 9. *$m(\vec{\mu}\vec{\mu}t) \equiv m(\vec{\mu}t)$ and $m(\vec{\mu}([t_2/x]t_1)) \equiv m([\vec{\mu}t_2/x]\vec{\mu}t_1)$*

Proof. We can prove this using the same method as lemma 3. We will not prove it here. □

Lemma 10. *If $a \to_\beta b$, then $m(a) \to_{\beta\mu} m(b)$.*

Proof. We prove this by induction on the derivation(depth) of $a \to_\beta b$. We list a few non-trial cases:

Base Case:

$$\frac{(x_i \mapsto t_i) \in \mu}{\mu x_i \to_\beta \mu t_i}$$

We have $m(\mu x_i) \equiv \mu x_i \to_{\beta\mu} m(\mu t_i)$.

Base Case:

$$\frac{}{(\lambda x.t)t' \to_\beta [t'/x]t}$$

We have $m((\lambda x.t)t') \equiv (\lambda x.m(t))m(t') \to_{\beta\mu} m([m(t)/x]m(t')) \equiv m([t'/x]t)$.

Step Case:

$$\frac{t \to_\beta t'}{\lambda x.t \to_\beta \lambda x.t'}$$

By IH, we have $m(\lambda x.t) \equiv \lambda x.m(t) \overset{IH}{\to}_{\beta\mu} \lambda x.m(t') \equiv m(\lambda x.t')$.

Step Case:

$$\frac{t \to_\beta t'}{\mu t \to_\beta \mu t'}$$

We want to show $m(\mu t) \to_{\beta\mu} m(\mu t')$. If $dom(\mu)\#FV(t)$, then $m(\mu t) \equiv m(t) \overset{IH}{\to}_{\beta\mu} m(t') \equiv m(\mu t')$. Of course, here we assume beta-reduction does not introduce any new variable.

If $dom(\mu) \cap FV(t) \neq \emptyset$, then identify t as $\dot{\vec{\mu}}_1 t''$, where t'' does not contain any closure at head position. We do case analyze on the structure of t'':

Case. $t'' = x_i \in dom(\dot{\vec{\mu}}_1)$ or $x_i \notin dom(\dot{\vec{\mu}}_1)$, these cases will not arise.

Case. $t'' = \lambda y.t_1$, then it must be that $t' = \dot{\overrightarrow{\mu_1}}(\lambda y.t_1')$ where $t_1 \to_\beta t_1'$. So we get $\mu\dot{\overrightarrow{\mu_1}}t_1 \to_\beta \mu\dot{\overrightarrow{\mu_1}}t_1'$. By IH(depth of $\mu\overrightarrow{\mu_1}t_1 \to_\beta \mu\dot{\overrightarrow{\mu_1}}t_1'$ is smaller), we have $m(\mu\overrightarrow{\mu_1}t_1) \to_{\beta\mu} m(\mu\overrightarrow{\mu_1}t_1')$. Thus $m(\mu\dot{\overrightarrow{\mu_1}}(\lambda y.t_1)) \equiv \lambda y.m(\mu\overrightarrow{\mu_1}t_1) \to_{\beta\mu} \lambda y.m(\mu\dot{\overrightarrow{\mu_1}}t_1') \equiv m(\mu\dot{\overrightarrow{\mu_1}}(\lambda y.t_1'))$.

Case. $t'' = t_1t_2$ and $t' = \dot{\overrightarrow{\mu_1}}(t_1't_2)$, where $t_1 \to_\beta t_1'$. We have $\mu\dot{\overrightarrow{\mu_1}}t_1 \to_\beta \mu\dot{\overrightarrow{\mu_1}}t_1'$. By IH(depth of $\mu\dot{\overrightarrow{\mu_1}}t_1 \to_\beta \mu\dot{\overrightarrow{\mu_1}}t_1'$ is smaller), $m(\mu\overrightarrow{\mu_1}t_1) \to_{\beta\mu} m(\mu\dot{\overrightarrow{\mu_1}}t_1')$. Thus $m(\mu\dot{\overrightarrow{\mu_1}}(t_1t_2)) \equiv m(\mu\dot{\overrightarrow{\mu_1}}t_1)m(\mu\dot{\overrightarrow{\mu_1}}t_2) \to_{\beta\mu} m(\mu\dot{\overrightarrow{\mu_1}}t_1')m(\mu\dot{\overrightarrow{\mu_1}}t_2) \equiv m(\mu\dot{\overrightarrow{\mu_1}}(t_1't_2))$. For $t'' = t_1t_2'$, where $t_2 \to_\beta t_2'$, we can argue similarly.

Case. $t'' = (\lambda y.t_1)t_2$ and $t' = \dot{\overrightarrow{\mu_1}}([t_2/y]t_1)$. $m(\mu\dot{\overrightarrow{\mu_1}}((\lambda y.t_1)t_2)) \equiv (\lambda y.m(\mu\dot{\overrightarrow{\mu_1}}t_1)))m(\mu\dot{\overrightarrow{\mu_1}}t_2) \to_{\beta\mu} m([m(\mu\dot{\overrightarrow{\mu_1}}t_2)/y]m(\mu\dot{\overrightarrow{\mu_1}}t_1)) \equiv m([\mu\dot{\overrightarrow{\mu_1}}t_2/y]\mu\dot{\overrightarrow{\mu_1}}t_1) \equiv m(\mu\dot{\overrightarrow{\mu_1}}[t_2/y]t_1)$(lemma 9).

□

Theorem 1. $\to_\beta \cup \to_\mu$ *is confluent.*

Proof. We know by diamond property of $\Rightarrow_{\beta\mu}$, $\to_{\beta\mu}$ is confluent. Since $\to_\mu$ is strongly normalizing and confluent, and by lemma 10 and Hardin's interpretation lemma(lemma 1), we conclude $\to_\beta \cup \to_\mu$ is confluent.

□

References

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