Confluence for Local Lambda-Mu Calculus

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Abstract
This note is taken directly from my comprehensive exam, modulo some re-organizations.

1 Hardin’s Interpretation Method

Sometimes it is inevitable to deal with reduction systems that contains more than one reduction, for example, $(\Lambda, \{\rightarrow_{\beta}, \rightarrow_{\eta}\})$. Confluence problem for this kind of system require some nontrivial efforts to prove. Hardin’s interpretation method [3] provide a way to deal with some of those reduction systems.

Lemma 1 (Interpretation lemma). Let $\rightarrow$ be $\rightarrow_1 \cup \rightarrow_2$, $\rightarrow_1$ being confluent and strongly normalizing. We denote by $\nu(a)$ the $\rightarrow_1$-normal form of $a$. Suppose that there is some relation $\rightarrow_i$ on $\rightarrow_1$ normal forms satisfying:

$\rightarrow_i \subseteq \rightarrow$, and $a \rightarrow_2 b$ implies $\nu(a) \rightarrow_i \nu(b)$ (†)

Then the confluence of $\rightarrow_i$ implies the confluence of $\rightarrow$.

Proof. So suppose $\rightarrow_i$ is confluent. If $a \rightarrow a'$ and $a \rightarrow a''$. So by (†), $\nu(a) \rightarrow_i \nu(a')$ and $\nu(a) \rightarrow_i \nu(a'')$. Notice that $t \rightarrow_{\hat{t}} t'$ implies $\nu(t) = \nu(t')$(By confluence and strong normalizing of $\rightarrow_1$). By confluence of $\rightarrow_i$, there exists $b$ such that $\nu(a') \rightarrow_i b$ and $\nu(a'') \rightarrow_i b$. Since $\rightarrow_i$, $\rightarrow_1 \subseteq \rightarrow$, we got $a' \rightarrow \nu(a') \rightarrow b$ and $a'' \rightarrow \nu(a'') \rightarrow b$. Hence $\rightarrow$ is confluent.

Hardin’s method reduce the confluence problem of $\rightarrow_1 \cup \rightarrow_2$ to $\rightarrow_i$, given the confluence and strong normalizing of $\rightarrow_1$, this make it possible to apply Tait-Martin-Löf’s (Takahashi’s) method to prove confluence of $\rightarrow_i$. 

□
2 Local $\lambda\mu$ Calculus

We now show an application of Hardin’s method on a concrete example, this example arise naturally in proving type preservation for Selfstar. The approach we adopt is similar to the one in [2].

Definition 1 (Local Lambda Mu Terms).

Terms $t ::= x \mid \lambda x.t \mid tt' \mid \mu t$

Closure $\mu ::= \{x_i \mapsto t_i\}_{i \in I}$

The closure is basically a set of recursively defined definitions. Let $I$ be a finite nonempty index set. For $\{x_i \mapsto t_i\}_{i \in I}$, we require for any $1 \leq i \leq n$, the set of free variables of $t_i$, $\text{FV}(t_i) \subseteq \text{dom}(\mu) = \{x_1, ..., x_n\}$ and we do not allow reduction, definition substitution, substitution inside the closure, we call it local property. Without this property, we are in the dangerous of losing confluence property (see [1] for a detailed discussion). $\mu \in t$ means the closure $\mu$ appears in $t$. $\mu t$ denotes $\mu_1...\mu_n t$. $\Phi$ denotes $\mu_1...\mu_n t$. $\mu t$ is strongly normalizing and confluent.

Definition 2 (Beta-Reductions).

$$\frac{(\lambda x.t)\mu \rightarrow \beta [t'/x]t}{x \in \text{FV}(t)} \quad \frac{t \rightarrow \beta t' \mu_i \rightarrow \beta \mu_i}{\lambda x.t \rightarrow \beta \lambda x.t} \quad \frac{tt' \rightarrow \beta t't' \mu \rightarrow \beta tt'}{t \rightarrow \beta t'} \quad \frac{t' \rightarrow \beta t''}{tt' \rightarrow \beta t't''} \quad \frac{t' \rightarrow \beta t''}{\mu t \rightarrow \beta \mu t'}$$

Definition 3 (Mu-Reductions).

$$\frac{\text{dom}(\mu)\neq \text{FV}(t)}{\mu t \rightarrow \mu t} \quad \frac{\mu(\lambda x.t) \rightarrow \beta \lambda x.t}{\mu(t_1t_2) \rightarrow \beta (\mu(t_1))t_2} \quad \frac{t \rightarrow \beta t'}{\lambda x.t \rightarrow \mu \lambda x.t}$$

2.0.1 Confluence of Local $\lambda\mu$ Calculus

Lemma 2. $\rightarrow_{\mu}$ is strongly normalizing and confluent.

Definition 4 ($\mu$-Normal Forms).

$$n ::= x \mid \mu x \mid \lambda x.n \mid mn'$$

We require $x_i \in \text{dom}(\mu)$.

Definition 5 ($\mu$-Normalize Function).

$$m(x) ::= x \quad m(\lambda y.t) ::= \lambda y.m(t)$$

$$m(t_1t_2) ::= m(t_1)m(t_2) \quad m(\mu y) ::= y \text{ if } y \notin \text{dom}(\mu)$$

$$m(\mu y) ::= \mu y \text{ if } y \in \text{dom}(\mu) \quad m(\mu(t')) ::= m(\mu(t))$$

$$m(\mu(\lambda x.t)) ::= \lambda x.m(\mu t)$$

Lemma 3. Let $\Phi$ denote the set of $\mu$ normal form, for any term $t$, $m(t) \in \Phi$.

Proof. One way to prove this is first identify $t$ as $\tilde{\mu}_1 t'$, here $\tilde{\mu}_1$ means there are zero or more closures and $t'$ does not contains any closure at head position. Then we can proceed by induction on the structure of $t'$:

Base Cases: $t' = x$, obvious.

Step Cases: If $t' = \lambda x.t''$, then $m(\tilde{\mu}_1(\lambda x.t'')) = \lambda x.m(\tilde{\mu}_1 t'')$. Now we can again identify $t''$ as $\tilde{\mu}_2 t'''$, where $t'''$ does not have any closure at head position. Since $t'''$ is structurally smaller than $\lambda x.t''$, by IH, $m(\tilde{\mu}_1\tilde{\mu}_2 t''') \in \Phi$, thus $m(\tilde{\mu}_1(\lambda x.t''')) \equiv \lambda x.m(\tilde{\mu}_1 t''') \in \Phi$.

For $t' = t_1 t_2$, we can argue similarly as above.

□
Lemma 5. If $\rightarrow\Rightarrow\text{Lemma 4.}$

By the lemma 5.

Proof. Lemma 7. By induction on the structure of Definition 6 by IH, $\Rightarrow\text{lemma 3: We identify Proof.}$

Note that the last three rules follows from the first rule. For the second one, because $n \rightarrow_\text{t}$ implies $\lambda x.n \rightarrow_\text{t} \lambda x.m(t)$ and $m(\lambda x.t) \equiv \lambda x.m(t)$. The others follow similarly.

Definition 6 ($\beta$ Reduction on $\mu$-normal Forms).

\[
\begin{align*}
\frac{n \rightarrow_\text{t} t}{n \rightarrow_{\beta_\mu} m(t)} & \quad \frac{n \rightarrow_{\beta_\mu} n'} {\lambda x.n \rightarrow_{\beta_\mu} \lambda x.n'} & \quad \frac{n' \rightarrow_{\beta_\mu} n''} {\lambda x.n \rightarrow_{\beta_\mu} \lambda x.n'} & \quad \frac{n \rightarrow_{\beta_\mu} n''} {\lambda x.n \rightarrow_{\beta_\mu} \lambda x.n'}
\end{align*}
\]

Note that the last three rules follows from the first rule. For the second one, because $n \rightarrow_\text{t}$ implies $\lambda x.n \rightarrow_\text{t} \lambda x.t$ and $m(\lambda x.t) \equiv \lambda x.m(t)$. The others follow similarly.

Definition 7 (Parallelization).

\[
\begin{align*}
\frac{n \Rightarrow_{\beta_\mu} n} {n \Rightarrow_{\beta_\mu} n} & \quad \frac{(x_i \rightarrow t_i) \in \mu \quad n_1 \Rightarrow_{\beta_\mu} n_1'} {\mu x_i \Rightarrow_{\beta_\mu} m(\mu x_i)}
\end{align*}
\]

Lemma 4. $\rightarrow_{\beta_\mu} \subseteq \Rightarrow_{\beta_\mu} \subseteq \rightarrow^*_\mu$.

Lemma 5. If $n_2 \Rightarrow_{\beta_\mu} n_2'$, then $m([n_2/x]n_1) \Rightarrow_{\beta_\mu} m([n_2'/x]n_1)$.

Proof. By induction on the structure of $n_1$. We list a few non-trivial cases:

Base Cases: $n_1 = x$, $n_1 = \mu x_i$, Obvious.

Step Case: $n_1 = \lambda y.n$. We have $m(\lambda y.[n_2/x]n) \equiv \lambda y.m([n_2/x]n) \Rightarrow_{\beta_\mu} \lambda y.m([n_2'/x]n) \equiv m(\lambda y.[n_2'/x]n)$.

Step Case: $n_1 = m n'$. We have $m([n_2/x]n[n_2/x]n') \equiv m([n_2/x]n)m([n_2/x]n') \Rightarrow_{\beta_\mu} m([n_2'/x]n)m([n_2'/x]n') \equiv m([n_2'/x]n[n_2'/x]n)$.

Lemma 6. $m(t) \equiv \beta_\mu m(t)$ and $m((m(t_1)/y)m(t_2)) \equiv m((t_1)/y)t_2)$.

Proof. The first equality is by lemma 3. For the second equality, we prove it through similar method as lemma 3: We identify $t_2$ as $\mu t_2$, $t_2'$ does not contains any closure at head position. We proceed by induction on the structure of $t_2'$

Base Cases: For $t_2' = x$, we use $m(t_2) \equiv m(t_2)$.

Step Cases: If $t_2' = \lambda x.t''$, then $m(\mu_{\lambda x}(\lambda x.[t_1/y]t''_2)) \equiv \lambda x.m(\mu_{\lambda x}(\lambda x.[t_1/y]t''_2)) \equiv \lambda x.m(\mu_{\lambda x}(\lambda x.[t_1/y]t''_2))$, where

Lemma 7. If $n_1 \Rightarrow_{\beta_\mu} n_1'$ and $n_2 \Rightarrow_{\beta_\mu} n_2'$, then $m([n_2/x]n_1) \Rightarrow_{\beta_\mu} m([n_2'/x]n_1)$.

Proof. We prove this by induction on the derivation of $n_1 \Rightarrow_{\beta_\mu} n_1'$.

Base Case:

$n \Rightarrow_{\beta_\mu} n$

By the lemma 5.
Base Case:

\[ x_i \mapsto t_i \in \mu \]
\[ \mu x_i \Rightarrow_{\beta \mu} m(\mu t_i) \]

Because \( y \notin \text{FV(}\mu x_i) \) and \( \mu \) is local.

Step Case:

\[ n_a \Rightarrow_{\beta \mu} n'_a \quad n_b \Rightarrow_{\beta \mu} n'_b \]
\[ (\lambda x. n_a) n_b \Rightarrow_{\beta \mu} m([n'_a/x]n'_b) \]

We have \( m((\lambda x. [n_2/y]n_a)[n_2/y]n_b) \equiv (\lambda x. m([n_2/y]n_a))m([n_2/y]n_b) \)
\[ \Rightarrow_{IH} \beta \mu m([m([n'_2/y]n'_b)/x]m([n'_2/y]n'_a)) \equiv m([n'_2/y]([n'_b/x]n'_a)). \]

The last equality is by lemma 6.

Step Case:

\[ n \Rightarrow_{\beta \mu} n' \]
\[ \lambda x. n \Rightarrow_{\beta \mu} \lambda x. n' \]

We have \( m((\lambda x. [n_2/y]n)[n_2/y]n_b) \equiv \lambda x. m([n_2/y]n) \Rightarrow_{IH} \lambda x. m([n'_2/y]n') \equiv m((\lambda x. [n'_2/y]n')) \)

Step Case:

\[ n_a \Rightarrow_{\beta \mu} n'_a \quad n_b \Rightarrow_{\beta \mu} n'_b \]
\[ n_an_b \Rightarrow_{\beta \mu} n'_an'_b \]

We have \( m((n_2/y)n_a[n_2/y]n_b) \equiv m([n_2/y]n_a)m([n_2/y]n_b) \)
\[ \Rightarrow_{IH} m([n'_2/y]n'_a)m([n'_2/y]n'_b) \equiv m([n'_2/y]([n'_a/x]n'_b)). \]

\[ \square \]

Lemma 8. If \( n \Rightarrow_{\beta \mu} n' \) and \( n \Rightarrow_{\beta \mu} n'' \), then there exist \( n''' \) such that \( n'' \Rightarrow_{\beta \mu} n''' \) and \( n' \Rightarrow_{\beta \mu} n''' \). So \( \Rightarrow_{\beta \mu} \) is confluent.

Proof. By induction on the derivation of \( n \Rightarrow_{\beta \mu} n' \).

Base Case:

\[ \overline{n} \Rightarrow_{\beta \mu} n \]

Obvious.

Base Case:

\[ \mu x_i \Rightarrow_{\beta \mu} m(\mu t_i) \]

Obvious.

Step Case:

\[ n_1 \Rightarrow_{\beta \mu} n'_1 \quad n_2 \Rightarrow_{\beta \mu} n'_2 \]
\[ (\lambda x. n_1)n_2 \Rightarrow_{\beta \mu} m([n'_1/x]n'_2) \]
Suppose \((\lambda x.n_1)n_2 \Rightarrow_\beta_\mu (\lambda x.n''_1)n''_2\), where \(n_1 \Rightarrow_\beta_\mu n'_1\) and \(n_2 \Rightarrow_\beta_\mu n''_2\). By lemma 7 and IH, we have 
\[m((n'_1/x)n''_2) \Rightarrow_\beta_\mu m((n''_1/x)n''_2)\]. We also have \((\lambda x.n''_1)n''_2 \Rightarrow_\beta_\mu m((n'''_1/x)n'''_2)\), where \(n''_1 \Rightarrow_\beta_\mu n''_1\) and \(n''_2 \Rightarrow_\beta_\mu n''_2\).

The other cases are either similar to the one above or easy.

One can also use Takahashi’s method([4]) to prove the lemma above. We will not explore that here.

**Lemma 9.** \(m(\mu\bar{\mu}t) \equiv m(\bar{\mu}t)\) and \(m(\mu([t_2/x]t_1)) \equiv m(\mu\bar{\mu}t_2/x)\).

**Proof.** We can prove this using the same method as lemma 3. We will not prove it here.

**Lemma 10.** If \(a \rightarrow_\beta b\), then \(m(a) \rightarrow_\beta m(b)\).

**Proof.** We prove this by induction on the derivation(depth) of \(a \rightarrow_\beta b\). We list a few non-trial cases:

**Base Case:**
\[(x_i \mapsto t_i) \in \mu \quad \mu x_i \rightarrow_\beta \mu t_i\]

We have \(m(\mu x_i) \equiv \mu x_i \rightarrow_\beta \mu m(t_i)\).

**Base Case:**
\[(\lambda x.t)' \rightarrow_\beta [t'/x]t\]

We have \(m((\lambda x.t)'') \equiv (\lambda x.m(t)m(t') \rightarrow_\beta \mu m([m(t)/x]m(t')) \equiv m([t'/x]t)\).

**Step Case:**
\[t \rightarrow_\beta t' \quad \lambda x.t \rightarrow_\beta \lambda x.t'\]

By IH, we have \(m(\lambda x.t) \equiv \lambda x.m(t) \rightarrow_\beta \mu \lambda x.m(t') \equiv m(\lambda x.t')\).

**Step Case:**
\[t \rightarrow_\beta t' \quad \mu t \rightarrow_\beta \mu t'\]

We want to show \(m(\mu t) \rightarrow_\beta \mu m(\mu t')\). If \(dom(\mu) \not\subseteq \text{FV}(t)\), then \(m(\mu t) \equiv m(t) \rightarrow_\beta \mu m(t') \equiv m(\mu t')\). Of course, here we assume beta-reduction does not introduce any new variable.

If \(dom(\mu) \cap \text{FV}(t) \neq \emptyset\), then identify \(t\) as \(\mu t''\), where \(t''\) does not contain any closure at head position. We do case analyze on the structure of \(t''\):

**Case.** \(t'' = x_i \in \text{dom}(\mu_1)\) or \(x_i \notin \text{dom}(\mu_1)\), these cases will not arise.

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Theorem 1. \( \rightarrow_{\beta} \cup \rightarrow_{\mu} \) is confluent.

Proof. We know by diamond property of \( \rightarrow_{\beta_{\mu}} \), \( \rightarrow_{\beta_{\mu}} \) is confluent. Since \( \rightarrow_{\mu} \) is strongly normalizing and confluent, and by lemma 10 and Hardin’s interpretation lemma (lemma 1), we conclude \( \rightarrow_{\beta} \cup \rightarrow_{\mu} \) is confluent.

References


