Type Preservation for Curry Style $\mathbf{F}_\omega$

Peng Fu
Computer Science, The University of Iowa

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1 System $\mathbf{F}_\omega$

Definition 1 (Syntax).
Terms $t ::= x \mid \lambda x.t \mid tt'$
Types $T ::= X \mid \Delta X : \kappa.T \mid T_1 \to T_2 \mid \lambda X.T \mid T_1 T_2$
Kinds $\kappa ::= * \mid \kappa' \to \kappa.$
Context $\Gamma ::= \cdot \mid \Gamma, x : T \mid \Gamma, X : \kappa$

Note: I use $\Delta X : \kappa.T$ to represent polymorphic functional types. I would have chose Π construct, but that will conflict Martin Löf's index functional types. So I choose Reynold's notation. The reason that I do not use $\forall$ is that it is a logical symbol, and it will blur the point of functional interpretation, namely, logic can be interpreted as part of the functional theory, but not the other way around.

Definition 2 (Well-formed Context).

$$\frac{}{\cdot \vdash \mathsf{wf}} \qquad \frac{\Gamma \vdash \mathsf{wf} \quad \Gamma \vdash T : *}{\Gamma, x : T \vdash \mathsf{wf}}$$

Definition 3 (Kinding).

$$\frac{(X : \kappa) \in \Gamma}{\Gamma \vdash X : \kappa}\ \mathit{KVar} \qquad \frac{\Gamma \vdash T_1 : * \quad \Gamma \vdash T_2 : *}{\Gamma \vdash T_1 \to T_2 : *}\ \mathit{Func} \qquad \frac{\Gamma, X : \kappa \vdash T : *}{\Gamma \vdash \Delta X : \kappa.T : *}\ \mathit{Poly}$$

$$\frac{\Gamma, X : \kappa \vdash T : \kappa'}{\Gamma \vdash \lambda X.T : \kappa \to \kappa'}\ \mathit{TAbs} \qquad \frac{\Gamma \vdash S : \kappa' \to \kappa \quad \Gamma \vdash T : \kappa'}{\Gamma \vdash ST : \kappa}\ \mathit{TApp}$$

Definition 4 (Typing Rules).

$$\frac{(x : T) \in \Gamma}{\Gamma \vdash x : T}\ \mathit{Var} \qquad \frac{\Gamma \vdash t : T_1 \quad \Gamma \vdash T_1 \simeq_\tau T_2 \quad \Gamma \vdash T_2 : *}{\Gamma \vdash t : T_2}\ \mathit{Conv} \qquad \frac{\Gamma, x : T_1 \vdash t : T_2 \quad \Gamma \vdash T_1 : *}{\Gamma \vdash \lambda x.t : T_1 \to T_2}\ \mathit{Func}$$

$$\frac{\Gamma \vdash t : T_1 \to T_2 \quad \Gamma \vdash t' : T_1}{\Gamma \vdash tt' : T_2}\ \mathit{App} \qquad \frac{\Gamma, X : \kappa \vdash t : T}{\Gamma \vdash t : \Delta X : \kappa.T}\ \mathit{Gen} \qquad \frac{\Gamma \vdash t : \Delta X : \kappa.T \quad \Gamma \vdash T' : \kappa}{\Gamma \vdash t : [T'/X]T}\ \mathit{Inst}$$

Note: $\simeq_\tau$ is the reflexive transitive and symmetry closure of $\rightarrowtail_\tau$.

Definition 5 (Type Reductions).

$$\frac{}{(\lambda X.T)T' \rightarrowtail_\tau [T'/X]T} \qquad \frac{T' \rightarrowtail_\tau T''}{TT' \rightarrowtail_\tau TT''} \qquad \frac{T \rightarrowtail_\tau T''}{TT' \rightarrowtail_\tau T''T'} \qquad \frac{T' \rightarrowtail_\tau T''}{T \to T' \rightarrowtail_\tau T \to T''}$$

$$\frac{T \rightarrowtail_\tau T''}{T \to T' \rightarrowtail_\tau T'' \to T'} \qquad \frac{T \rightarrowtail_\tau T'}{\Delta X : \kappa.T \rightarrowtail_\tau \Delta X : \kappa.T'} \qquad \frac{T \rightarrowtail_\tau T'}{\lambda X.T \rightarrowtail_\tau \lambda X.T'}$$

Definition 6 (Term Reductions).

$$\frac{}{(\lambda x.t)t' \to_\beta [t'/x]t} \qquad \frac{t' \to_\beta t''}{tt' \to_\beta tt''} \qquad \frac{t \to_\beta t''}{tt' \to_\beta t''t'} \qquad \frac{t \to_\beta t'}{\lambda x.t \to_\beta \lambda x.t'}$$

2 Morph Analysis

Definition 7 (Morphing)**.**

- $T_1 \rightarrowtail_i T_2$ *if* $T_1 \equiv \Delta X : \kappa.T$ *and* $T_2 \equiv [T'/X]T$ *for some* T'.
- $T_1 \rightarrowtail_g T_2$ *if* $T_2 \equiv \Delta X : \kappa.T_1$ *for some* X, κ.

Let $\rightarrowtail_{i,g}^*$ be the reflexive and transitive closure of $\rightarrowtail_i \cup \rightarrowtail_g$. Let $\overline{\Gamma} = \Gamma, X_1 : \kappa_1, ..., X_n : \kappa_n$ for some $\{X_i : \kappa_i\}_{i \in N}$.

Definition 8.
$$o(\Delta X : \kappa.T) := o(T) \quad o(X) := X \quad o(T_1 \to T_2) := T_1 \to T_2 \quad o(\lambda X.T) := \lambda X.T \quad o(T_1 T_2) := T_1 T_2$$

Lemma 1. $o([T/X]T') \equiv [T''/X]o(T')$ *for some* T''.

Proof. By induction on structure of T'. □

Note: We do not have: if $T =_\beta T'$, then $o(T) =_\beta o(T')$. Due to the fact that we do not have $o([T_1/X]T_2) \equiv [o(T_1)/X]o(T_2)$.

Lemma 2. *If* $T \rightarrowtail_{i,g}^* T'$, *then there exist a type substitution* σ *such that* $\sigma o(T) \equiv o(T')$.

Proof. It suffices to consider $T \rightarrowtail_{i,g} T'$. If $T' \equiv \Delta X : \kappa.T$, then $o(T') \equiv o(T)$. If $T \equiv \Delta X : \kappa.T_1$ and $T' \equiv [T''/X]T_1$, then $o(T) \equiv o(T_1)$. By lemma 1 above, we know $o(T') \equiv o([T''/X]T_1) \equiv [T_2/X]o(T_1)$ for some T_2. □

Lemma 3. *If* $T_1 \to T_2 \rightarrowtail_{i,g}^* T_1' \to T_2'$, *then there exist a type substitution* σ *such that* $\sigma(T_1 \to T_2) \equiv T_1' \to T_2'$.

Proof. By lemma 2 above. □

Let $\rightarrowtail_{i,g,\tau}^*$ denotes $(\rightarrowtail_{i,g}^* \cup \simeq_\tau)^*$.

Theorem 1 (Compatibility)**.** *If* $T_1 \to T_2 \rightarrowtail_{i,g,\tau}^* T_1' \to T_2'$, *then there exist a type substitution* σ *such that* $\sigma(T_1 \to T_2) \simeq_\tau T_1' \to T_2'$.

3 Type Preservation

Lemma 4. *Let* $T_1 \rightarrowtail_{i,g,\tau}^* T_2$. *Then* $\Gamma \vdash t : T_1$ *implies* $\Gamma_1 \vdash t : T_2$, *where* $\Gamma = \overline{\Gamma}_1$.

Lemma 5 (Inversion I)**.** *If* $\Gamma \vdash x : T$, *then exist* Γ_1, T_1 *such that* $T_1 \rightarrowtail_{i,g,\tau}^* T$ *and* $(x : T_1) \in \Gamma_1$ *and* $\Gamma_1 = \overline{\Gamma}$.

Lemma 6 (Inversion II)**.** *If* $\Gamma \vdash t_1 t_2 : T$, *then exist* Γ_1, T_1, T_2 *such that* $\Gamma_1 \vdash t_1 : T_1 \to T_2$ *and* $\Gamma_1 \vdash t_2 : T_1$ *and* $T_2 \rightarrowtail_{i,g,\tau}^* T$ *with* $\Gamma_1 = \overline{\Gamma}$.

Lemma 7 (Inversion III)**.** *If* $\Gamma \vdash \lambda x.t : T$, *then exist* Γ_1, T_1, T_2 *such that* $\Gamma_1, x : T_1 \vdash t : T_2$ *and* $T_1 \to T_2 \rightarrowtail_{i,g,\tau}^* T$ *with* $\Gamma_1 = \overline{\Gamma}$.

Lemma 8 (Substitution)**.**

1. *If* $\Gamma \vdash t : T$, *then for any type substitution* σ, $\sigma\Gamma \vdash t : \sigma T$.
2. *If* $\Gamma, x : T \vdash t : T'$ *and* $\Gamma \vdash t' : T$, *then* $\Gamma \vdash [t'/x]t : T'$.

Proof. By induction on derivation. □

Theorem 2 (Type Preservation)**.** *If* $\Gamma \vdash t : T$ *and* $t \to_\beta t'$, *then* $\Gamma \vdash t' : T$.

Proof. By induction on derivation of $\Gamma \vdash t : T$. We only show one interesting case here:

$$\frac{\Gamma \vdash t : T_1 \to T_2 \quad \Gamma \vdash t' : T_1}{\Gamma \vdash tt' : T_2}\ App$$

Suppose $(\lambda x.t_1)t_2 \to_\beta [t_2/x]t_1$. We know that $\Gamma \vdash \lambda x.t_1 : T_1 \to T_2$ and $\Gamma \vdash t_2 : T_1$. By inversion on $\Gamma \vdash \lambda x.t_1 : T_1 \to T_2$, we know that there exist Γ_1, T_1', T_2' such that $\Gamma_1, x : T_1' \vdash t_1 : T_2'$ and $T_1' \to T_2' \rightarrowtail^*_{i,g,\tau} T_1 \to T_2$ with $\Gamma_1 = \overline{\Gamma}$. By theorem 1, we have $\sigma(T_1' \to T_2') \simeq_\tau T_1 \to T_2$. By Church-Rosser of $\simeq_\tau$, we have $\sigma T_1' \simeq_\tau T_1$ and $\sigma T_2' \simeq_\tau T_2$. So by (1) of lemma 8, we have $\Gamma_1, x : \sigma T_1' \vdash t_1 : \sigma T_2'$. Thus $\Gamma, x : T_1 \vdash t_1 : T_2$. Since $\Gamma \vdash t_2 : T_1$. By (2) of lemma 8, $\Gamma \vdash [t_2/x]t_1 : T_2$. □