The Reducibility Method for Call-By-Value Simply Typed Lambda Calculus

Reorganized by Peng Fu

Last revised: March 28,2010

1 Descriptions

1.1 Types

$$T \ ::= b |\ T_1 \to T_2$$

1.2 Terms.

$$t \ ::= \ x \mid (t_1 \ t_2) | \lambda x.t$$

1.3 Type assignment rules.

$$\frac{\Gamma(x) = T}{\Gamma \vdash x : T} \ T\_Var$$

$$\frac{\Gamma \vdash t_1 : T_2 \to T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash t_1 \ t_2 : T_1} \ T\_App$$

$$\frac{\Gamma, x : T_1 \ \vdash t : T_2}{\Gamma \vdash \lambda x.t : T_1 \to T_2} \ T\_Lam$$

1.4 Reduction rules.

Left-to-right,call-by-value reduction.
Contexts.

$$C \ ::= * |\ v\ C \mid C\ t$$

Values.

$$v \ ::= \lambda x.t$$

Reduction.

$$C[(\lambda x.t \ v)] \leadsto C[[v/x]t]$$

2 Reducibility

2.1 Properties of Reduciblity sets

Definition Let N be the set of terms which have a normal form under our reduction setting. We define sets RED_T by induction on T

1. $t \in RED_b$ iff $t \in N$ and closed.

2. $t \in RED_{T_1 \to T_2}$ iff $\forall u\ (u \in RED_{T_1} \Rightarrow (t\ u) \in RED_{T_2})$.

CR 1 If $t \in RED_T$, then $t \in N$ *and closed.*

CR 2 If $t \in RED_T$ *and* $t \leadsto t'$, then $t' \in RED_T$.

CR 3 If t is a closed term, $t \leadsto t'$ *and* $t' \in RED_T$, then $t \in RED_T$.

CR 4 RED_T is a non-empty set.

Proof We will do the induction on the structure of T:

Base Case: $T = b$

(CR 1) is a tautology.

(CR 2) By definition of RED_b,if $t \in RED_b$, then $t \in N$ *and closed.* Since reduction is deterministic, $t \leadsto t'$ *and* $t \in N$ implies $t' \in N$. Also, the reduction cannot introduce new free variable, so t closed implies t' closed. So $t' \in RED_b$.

(CR 3) By definition of N, $t \leadsto t', t' \in N$ implies $t \in N$. Since t is closed by assumption, $t \in RED_b$.

(CR 4) Obvious.

Step Case: $T = T_1 \to T_2$

(CR 1) Assume $t \in RED_{T_1 \to T_2}$. By IH(CR 4), RED_{T_1} is non-empty. So let u be an arbitrary element of RED_{T_1}. Now by the definition of $RED_{T_1 \to T_2}$, $(t\ u) \in RED_{T_2}$. By IH(CR 1), $u \in N$ *and closed*, $(t\ u) \in N$ *and closed*, which implies t is also closed. We need to show $t \in N$. Let $\nu(t\ u)$ denoted the length of the reduction from $(t\ u)$ to its normal form, the proof is by induction on $\nu(t\ u)$:

Base Case: $\nu(t\ u) = 0$, but this case cannot arise, since $\nu(t\ u) = 0$ implies t is a variable, but we know t is closed.

Step Case: $(t\ u)$ can be further reduced, if the call-by-value redex is in t, then $(t\ u) \leadsto (t'\ u)$, and by IH, $t' \in N$,so $t \in N$. If the redex is in u,that means t contain no call-by-value redex,so $t \in N$. If the whole $(t\ u)$ is a redex, then t must be a lambda term, which is a normal form under the call-by-value reduction, so $t \in N$.

So $t \in N$ *and closed.*

(CR 2) Assume $t \in RED_{T_1 \to T_2}$. Let u be an arbitrary element of RED_{T_1}. Now by definition of $RED_{T_1 \to T_2}$, we have $(t\ u) \in RED_{T_2}$. And since $t \leadsto t'$, by definition of left-to-right, call-by-value reduction, we have the reduction: $(t\ u) \leadsto (t'\ u)$. By IH(CR 2), we have $(t'\ u) \in RED_{T_2}$, so according to the definition of $RED_{T_1 \to T_2}$, $t' \in RED_{T_1 \to T_2}$.

(CR 3) Suppose t is closed and $t \leadsto t'$, and $t' \in RED_{T_1 \to T_2}$. Let u be an arbitrary element of RED_{T_1}. By definition of $RED_{T_1 \to T_2}$, $(t'\ u) \in RED_{T_2}$. Now let's consider $(t\ u)$. By definition of left-to-right, call-by-value, the only reduction it can have is $(t\ u) \leadsto (t'\ u)$, and we already know $(t'\ u) \in RED_{T_2}$. Also

$(t\ u)$ is closed by assumption and IH(CR 1). By IH(CR 3), we have $(t\ u) \in\ RED_{T_2}$. Then by definition of $RED_{T_1 \to T_2}$, we have $t \in\ RED_{T_1 \to\ T_2}$.

(CR 4) We need to show $RED_{T_1 \to T_2}$ is non-empty. By IH(CR 4), both RED_{T_1} *and* RED_{T_2} are non-empty. So it suffices to show $\lambda x.t \in RED_{T_1 \to T_2}, where\ t \in RED_{T_2}$. For arbitary $u \in RED_{T_1}$, by IH(CR 1), $u \in N\ and\ closed$. By the definition of left-to-right, call-by-value reduction, we have $(\lambda x.t)\ u \overset{*}{\rightsquigarrow} (\lambda x.t)\ v \rightsquigarrow t$. Because $t \in RED_{T_2}$,and $(\lambda x.t)\ u$ is closed, by IH(CR 3), $(\lambda x.t)\ u \in RED_{T_2}$. So by the definition of $RED_{T_1 \to T_2}$, $\lambda x.t \in RED_{T_1 \to T_2}$. So $RED_{T_1 \to T_2}$ is non-empty.

2.1.1 Reducibility and Type assignment

Definition We define the set $[\Gamma]$ of well-typed substitutions σ as follows:

$$\frac{}{\Phi \in [.]}$$

$$\frac{\sigma \in [\Gamma] \quad t \in RED_T}{\sigma \cup \{(x,t)\} \in [\Gamma, x : T]}$$

Theorem If $\Gamma \vdash t : T$, then $\forall \sigma \in [\Gamma], \sigma\ t \in RED_T$.

Proof By induction on the typing derivation of $\Gamma \vdash t : T$

Base Case The typing derivation looks like:

$$\frac{\Gamma(x) = T}{\Gamma \vdash x : T}$$

By definition of σ, for any $\sigma \in [\Gamma]$,then $\{(x,t)\} \subseteq \sigma, t \in RED_T$, so $\sigma\ x = t \in RED_T$.

Application Case The typing derivation looks like:

$$\frac{\Gamma \vdash t_1 : T_2 \to T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash t_1\ t_2 : T_1}$$

We need to prove that $\sigma(t_1\ t_2) \in RED_{T_1}$. By IH, for any $\sigma \in [\Gamma], \sigma\ t_1 \in\ RED_{T_2 \to T_1}\ and\ \sigma\ t_2 \in\ RED_{T_2}$. Then from defintion of $RED_{T_2 \to T_1}$, we have $(\sigma t_1)(\sigma t_2) = \sigma(t_1\ t_2) \in RED_{T_1}$.

Lambda abstract Case The typing derivation look like:

$$\frac{\Gamma, x : T_1\ \vdash t : T_2}{\Gamma \vdash \lambda x.t : T_1 \to T_2}$$

We need to show any $\sigma \in [\Gamma]$, we have $\sigma(\lambda x.t) = \lambda x.(\sigma\ t) \in RED_{T_1 \to T_2}$. By definition of $RED_{T_1 \to T_2}$,we need to show for arbitrary $u \in RED_{T_1}, (\lambda x.(\sigma\ t))\ u \in RED_{T_2}$. Since u is closed by CR 1, the normal form of u must be a value, which means $u \overset{*}{\rightsquigarrow} v$. So we have $(\lambda x.(\sigma\ t))\ u \overset{*}{\rightsquigarrow} (\lambda x.(\sigma\ t))\ v$, and by CR 2, $v \in RED_{T_1}$. By definition of call-by-value reduction, $(\lambda x.(\sigma\ t))\ v \rightsquigarrow \sigma[v/x]t$. Since $v \in RED_{T_1}$,$\sigma \cup \{(x,v)\} \in [\Gamma, x : T_1]$. By IH, $\sigma[v/x](t) \in RED_{T_2}$. Since $(\lambda x.(\sigma\ t))\ u$ is closed ,by CR 3, $(\lambda x.(\sigma\ t))\ u \in RED_{T_2}$. So $\sigma(\lambda x.t) = \lambda x.(\sigma\ t) \in RED_{T_1 \to T_2}$.

3 Conclusion

So for any closed term t, if $\vdash t : T$, then $t \in RED_T$, and by CR 1,$t \in N$.