The Reducibility Method for Call-By-Value Simply Typed Lambda Calculus

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1 Descriptions

1.1 Types
\[ T ::= b \mid T_1 \to T_2 \]

1.2 Terms.
\[ t ::= x \mid (t_1 \cdot t_2)\lambda x.t \]

1.3 Type assignment rules.
\[
\begin{align*}
\Gamma(x) &= T \quad T_{Var} \\
\Gamma \vdash x : T &\quad T_{Var} \\
\Gamma \vdash t_1 : T_2 \to T_1 \quad \Gamma \vdash t_2 : T_2 &\quad T_{App} \\
\Gamma, x : T_1 \vdash t : T_2 &\quad T_{Lam}
\end{align*}
\]

1.4 Reduction rules.
Left-to-right, call-by-value reduction.

Contexts.
\[ C ::= * \mid v \mid C \cdot t \]

Values.
\[ v ::= \lambda x.t \]

Reduction.
\[ C[(\lambda x.t \cdot v)] \rightsquigarrow C[[v/x][t]] \]

2 Reducibility

2.1 Properties of Reducibility sets

Definition Let \( N \) be the set of terms which have a normal form under our reduction setting. We define sets \( RED_T \) by induction on \( T \).
1. \( t \in \text{RED}_b \) iff \( t \in \mathbb{N} \) and closed.

2. \( t \in \text{RED}_{T_1 \rightarrow T_2} \) iff \( \forall u \ (u \in \text{RED}_{T_1} \Rightarrow (t \ u) \in \text{RED}_{T_2}) \).

**CR 1** If \( t \in \text{RED}_T \), then \( t \in \mathbb{N} \) and closed.

**CR 2** If \( t \in \text{RED}_T \) and \( t \leadsto t' \), then \( t' \in \text{RED}_T \).

**CR 3** If \( t \) is a closed term, \( t \leadsto t' \) and \( t' \in \text{RED}_T \), then \( t \in \text{RED}_T \).

**CR 4** \( \text{RED}_T \) is a non-empty set.

**Proof** We will do the induction on the structure of \( T \):

**Base Case:** \( T = b \)

(CR 1) is a tautology.

(CR 2) By definition of \( \text{RED}_b \), if \( t \in \text{RED}_b \), then \( t \in \mathbb{N} \) and closed. Since reduction is deterministic, \( t \leadsto t' \) and \( t \in \mathbb{N} \) implies \( t' \in \mathbb{N} \). Also, the reduction cannot introduce new free variable, so \( t \) closed implies \( t' \) closed. So \( t' \in \text{RED}_b \).

(CR 3) By definition of \( \mathbb{N} \), \( t \leadsto t', t' \in \mathbb{N} \) implies \( t \in \mathbb{N} \). Since \( t \) is closed by assumption, \( t \in \text{RED}_b \).

(CR 4) Obvious.

**Step Case:** \( T = T_1 \rightarrow T_2 \)

(CR 1) Assume \( t \in \text{RED}_{T_1 \rightarrow T_2} \). By IH(CR 4), \( \text{RED}_{T_1} \) is non-empty. So let \( u \) be an arbitrary element of \( \text{RED}_{T_1} \). Now by the definition of \( \text{RED}_{T_1 \rightarrow T_2} \), \( (t \ u) \in \text{RED}_{T_2} \). By IH(CR 1), \( u \in \mathbb{N} \) and closed, \( (t \ u) \in \mathbb{N} \) and closed, which implies \( t \) is also closed. We need to show \( t \in \mathbb{N} \). Let \( \nu(t \ u) \) denoted the length of the reduction from \( (t \ u) \) to its normal form, the proof is by induction on \( \nu(t \ u) \):

**Base Case:** \( \nu(t \ u) = 0 \), but this case cannot arise, since \( \nu(t \ u) = 0 \) implies \( t \) is a variable, but we know \( t \) is closed.

**Step Case:** \( (t \ u) \) can be further reduced, if the call-by-value redex is in \( t \), then \( (t \ u) \leadsto (t' \ u) \), and by IH, \( t' \in \mathbb{N} \), so \( t \in \mathbb{N} \). If the redex is in \( u \), that means \( t \) contain no call-by-value redex, so \( t \in \mathbb{N} \). If the whole \( (t \ u) \) is a redex, then \( t \) must be a lambda term, which is a normal form under the call-by-value reduction, so \( t \in \mathbb{N} \).

So \( t \in \mathbb{N} \) and closed.

(CR 2) Assume \( t \in \text{RED}_{T_1 \rightarrow T_2} \). Let \( u \) be an arbitrary element of \( \text{RED}_{T_1} \). Now by definition of \( \text{RED}_{T_1 \rightarrow T_2} \), we have \( (t \ u) \in \text{RED}_{T_2} \). And since \( t \leadsto t' \), by definition of left-to-right, call-by-value reduction, we have the reduction: \( (t \ u) \leadsto (t' \ u) \). By IH(CR 2), we have \( (t' \ u) \in \text{RED}_{T_2} \), so according to the definition of \( \text{RED}_{T_1 \rightarrow T_2} \), \( t' \in \text{RED}_{T_1 \rightarrow T_2} \).

(CR 3) Suppose \( t \) is closed and \( t \leadsto t' \), and \( t' \in \text{RED}_{T_1 \rightarrow T_2} \). Let \( u \) be an arbitrary element of \( \text{RED}_{T_1} \). By definition of \( \text{RED}_{T_1 \rightarrow T_2} \), \( (t' \ u) \in \text{RED}_{T_2} \). Now let’s consider \( (t \ u) \). By definition of left-to-right, call-by-value, the only reduction it can have is \( (t \ u) \leadsto (t' \ u) \), and we already know \( (t' \ u) \in \text{RED}_{T_2} \). Also
(t u) is closed by assumption and IH(CR 1). By IH(CR 3), we have (t u) ∈ RED₁₂. Then by definition of RED₁→₁₂, we have t ∈ RED₁₁→₁₂.

(CR 4) We need to show RED₁₁→₁₂ is non-empty. By IH(CR 4), both RED₁₁ and RED₁₂ are non-empty. So it suffices to show λx.t ∈ RED₁₁→₁₂, where t ∈ RED₁₂. For arbitrary u ∈ RED₁₁, by IH(CR 1), u ∈ N and closed. By the definition of left-to-right, call-by-value reduction, we have (λx.t) u ≈ (λx.t) v ∈ T. Because t ∈ RED₁₂, and (λx.t) u is closed, by IH(CR 3), (λx.t) u ∈ RED₁₂. So by the definition of RED₁₁→₁₂, λx.t ∈ RED₁₁→₁₂. So RED₁₁→₁₂ is non-empty.

2.1.1 Reducibility and Type assignment

Definition We define the set [Γ] of well-typed substitutions σ as follows:

\[ \Phi \in [\Gamma] \]

\[ \sigma \in [\Gamma] \quad t \in RED_T \quad \sigma \cup \{(x,t)\} \in [\Gamma, x:T] \]

Theorem If \( \Gamma \vdash t : T \), then \( \forall \sigma \in [\Gamma], \sigma t \in RED_T \).

Proof By induction on the typing derivation of \( \Gamma \vdash t : T \).

Base Case The typing derivation looks like:

\( \Gamma(x) = T \)
\( \Gamma \vdash x : T \)

By definition of σ, for any \( \sigma \in [\Gamma] \), then \( \{(x,t)\} \subseteq \sigma, t \in RED_T \), so \( \sigma x = t \in RED_T \).

Application Case The typing derivation looks like:

\( \Gamma \vdash t_1 : T_2 \rightarrow T_1 \quad \Gamma \vdash t_2 : T_2 \quad \Gamma \vdash t_1 t_2 : T_1 \)

We need to prove that \( \sigma(t_1 t_2) \in RED_{T_1} \). By IH, for any \( \sigma \in [\Gamma], \sigma t_1 \in RED_{T_2 \rightarrow T_1} \) and \( \sigma t_2 \in RED_{T_2} \). Then from definition of RED₁₂→₁₁, we have \( \sigma(t_1)\sigma(t_2) = \sigma(t_1 t_2) \in RED_{T_1} \).

Lambda abstract Case The typing derivation looks like:

\( \Gamma, x : T_1 \vdash t : T_2 \quad \Gamma \vdash \lambda x.t : T_1 \rightarrow T_2 \)

We need to show any \( \sigma \in [\Gamma] \), we have \( \sigma(\lambda x.t) = \lambda x.(\sigma t) \in RED_{T_1 \rightarrow T_2} \). By definition of RED₁₁→₁₂, we need to show for arbitrary u ∈ RED₁₁, \( (\lambda x.(\sigma t)) u \in RED_{T_2} \). Since u is closed by CR 1, the normal form of u must be a value, which means u ∼ v. So we have \( (\lambda x.(\sigma t)) u \sim \lambda x.(\sigma t) v \), and by CR 2, \( v \in RED_{T_1} \). By definition of call-by-value reduction, \( (\lambda x.(\sigma t)) v \sim \sigma[v/x]t \). Since \( v \in RED_{T_1}, \sigma \cup \{(x,v)\} \in [\Gamma, x : T_1] \), By IH, \( \sigma[v/x](t) \in RED_{T_2} \). Since \( (\lambda x.(\sigma t)) u \) is closed, by CR 3, \( (\lambda x.(\sigma t)) u \in RED_{T_2} \). So \( \sigma(\lambda x.t) = \lambda x.(\sigma t) \in RED_{T_1 \rightarrow T_2} \).

3 Conclusion

So for any closed term t, if \( \Gamma \vdash t : T \), then t ∈ REDₜ, and by CR 1, t ∈ N.