Subtyping Relation as Reducibility Set Inclusion

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1 Descriptions

1.1 Types

$$T \ ::= b | \ T_1 \to T_2$$

1.2 Terms.

$$t \ ::= \ x \ | \ (t_1 \ t_2) | \lambda x.t$$

1.3 Type assignment rules.

$$\frac{\Gamma(x) = T}{\Gamma \vdash x : T} \ T\_Var$$

$$\frac{\Gamma \vdash t_1 : T_2 \to T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash t_1 \ t_2 : T_1} \ T\_App$$

$$\frac{\Gamma, x : T_1 \ \vdash t : T_2}{\Gamma \vdash \lambda x.t : T_1 \to T_2} \ T\_Lam$$

$$\frac{\Gamma \vdash t : T_1 \quad T_1 <: T_2}{\Gamma \vdash t : T_2} \ T\_Sub$$

1.4 Subtyping

$$\frac{}{T <: b} \ Top$$

$$\frac{T_1' <: T_1 \quad T_2 <: T_2'}{T_1 \to T_2 <: T_1' \to T_2'} \ Coercion$$

2 Interpretation

2.1 Properties of Reduciblity Sets

Definition Let N be the set of terms which have a normal form under our reduction setting. We define sets RED_T by induction on T

1. $t \in RED_b$ iff $t \in N$ and closed.

2. $t \in RED_{T_1 \to \ T_2}$ iff $\forall u \ (u \in \ RED_{T_1} \ \Rightarrow \ (t \ u) \in \ RED_{T_2})$.

CR 1 If $t \in RED_T$, then $t \in N$ *and closed.*

CR 2 If $t \in RED_T$ *and* $t \rightsquigarrow t'$, then $t' \in RED_T$.

CR 3 If t is a closed term, $t \rightsquigarrow t'$ *and* $t' \in RED_T$, then $t \in RED_T$.

CR 4 RED_T is a non-empty set.

Theorem(Soundness of Subtyping) If $t \in RED_T$ and $T <: T'$, then $t \in RED_{T'}$.

Proof: We will do the induction on the structure of T:

Base Case: $T = b$

By inversion on subtyping relation, $T' = b$, so it's trivially true.

Step Case: $T = T_1 \rightarrow T_2$

If $T' = b$, $t \in RED_T$, by CR 1, $t \in N$, so $t \in RED_b$. If $T' = T_1' \rightarrow T_2'$, by inversion on the subtyping relation, $T_1' <: T_1$ *and* $T_2 <: T_2'$. So for arbitrary $u \in RED_{T_1'}$, by IH, $u \in RED_{T_1}$. By definition of $RED_{T_1 \rightarrow T_2}$, $tu \in RED_{T_2}$. Again, by IH, $tu \in RED_{T_2'}$. So by definition of $RED_{T_1' \rightarrow T_2'}$, $t \in RED_{T_1' \rightarrow T_2'}$.

3 Conclusion

If we interpret type T as RED_T, then we can interpret subtyping relation $T <: T'$ as $RED_T \subseteq RED_{T'}$.