The Reducibility Method for Call-By-Value System F

Reorganized by Peng Fu

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1 Descriptions

1.1 Types

$T ::= X \mid T_1 \to T_2 \mid \Pi X.T$

1.2 Terms

$t ::= x \mid (t_1\ t_2) \mid \lambda x.t \mid c$

1.3 Well-formed Context Γ

Let $FV(T)$ denote all the free variables of the type T, $dom(\Gamma)$ denote the type and term variables in the context Γ.

$\Gamma ::= \cdot \mid \Gamma, X : type \mid \Gamma, x : T.$

$$\frac{}{\emptyset\ OK}$$

$$\frac{\Gamma\ OK}{\Gamma, X : type\ OK}$$

$$\frac{\Gamma\ OK\ FV(T) \subseteq dom(\Gamma)}{\Gamma, x : T\ OK}$$

A context Γ is well-formed iff $\Gamma\ OK$. We assume all contexts in this note are well-formed.

1.4 Type assignment rules.

$$\frac{\Gamma(x) = T}{\Gamma \vdash x : T}\ T\_Var$$

$$\frac{\Gamma \vdash t_1 : T_2 \to T_1 \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash t_1\ t_2 : T_1}\ T\_App$$

$$\frac{\Gamma, x : T_1\ \vdash t : T_2}{\Gamma \vdash \lambda x.t : T_1 \to T_2}\ T\_Lam$$

$$\frac{\Gamma, X : type\ \vdash t : T}{\Gamma \vdash t : \Pi X.T}\ Univ\_Abs$$

$$\frac{\Gamma\ \vdash t : \Pi X.T}{\Gamma \vdash t : [U/X]T}\ Univ\_App$$

1.5 Reduction rules

Left-to-right,call-by-value reduction.

Contexts

$C \ ::= * | \ v\ C \mid C\ t$

Values

$v \ ::= \lambda x.t \mid i$

Inactive terms

$i \ ::= c \mid (i\ v)$

Reductions

$C[(\lambda x.t)\ v] \rightsquigarrow C[[v/x]t]$

2 Reducibility

2.1 Reducibility Candidates

Let N be the set of terms which have a normal form under our reduction setting. Let I be the set of all inactive terms.

Definition A reducibility candidate R is a set of terms that satisfies the following conditions:

CR 1 If $t \in R$,then $t \in N$ *and closed*.

CR 2 If $t \in R$ *and* $t \rightsquigarrow t'$, then $t' \in R$.

CR 3 If t is a closed term, $t \rightsquigarrow t'$ *and* $t' \in R$, then $t \in R$.

CR 4 $I \subseteq R$.

Fact Let $\Re$ be the set of all reducibility candidates. $\Re$ is a non-empty set.

To show $\Re$ is non-empty, we will just show all the closed term $t \in N$ is a reducibility candidate, which is obvious from the definition of reducibility candidate.

2.2 Reducibility Sets

As Girard said: Among all the *candidates*, the *true reducibility candidate* for T is to be found.Here we use *reducibility set* to find the right candidate.

Definition Let ϕ be a finite function: $FV(T) \rightarrow \Re$. Eg. $\phi(X) \in \Re$. If $dom(\phi) = \{X_1, X_2, ...X_n\}$, then we usually write ϕ as $[R_1/X_1, ...R_n/X_n]$.

The reducibility set $RED_T\phi$ is defined inductively as follows.

$t \in RED_X\phi$ iff $t \in \phi(X)$.

$t \in RED_{T_1 \to T_2}\phi$ iff $(\forall u \in RED_{T_1}\phi \Rightarrow (t\ u) \in RED_{T_2}\phi)$.

$t \in RED_{\Pi Y.W}\phi$ iff $\forall R \in \Re, t \in RED_W\phi[R/Y]$.

2.3 Reducibility Sets as Reducibility Candidates

Now we can show that our reducibility sets are indeed reducibility candidates.

CR 1 If $t \in RED_T\phi$, then $t \in N$ and closed.

CR 2 If $t \in RED_T\phi$ *and* $t \rightsquigarrow t'$, then $t' \in RED_T\phi$.

CR 3 If t is a closed term, $t \rightsquigarrow t'$ *and* $t' \in RED_T\phi$, then $t \in RED_T\phi$.

CR 4 $I \subseteq RED_T\phi$.

Proof By induction on the structure of T.

Base Case: $T = X$

CR 1–CR 4 Obvious from the definition.

Step Case: $T = T_1 \to T_2$

CR 1 Assume $t \in RED_{T_1 \to T_2}\phi$. By IH(CR 4), $RED_{T_1}\phi$ is non-empty. So we can take arbitrary $u \in RED_{T_1}\phi$. By definition, $(t\ u) \in RED_{T_2}\phi$. By IH(CR 1), $(t\ u) \in N$, $u \in N$. So $t \in N$.

CR 2 Assume $t \in RED_{T_1 \to T_2}\phi$ and $t \rightsquigarrow t'$. Take arbitrary $u \in RED_{T_1}\phi$. By definition, we know $(t\ u) \in RED_{T_2}\phi$. With our reduction strategy, $(t\ u) \rightsquigarrow (t'\ u)$. By IH(CR 2), $(t'\ u) \in RED_{T_2}\phi$. So by definition of $RED_{T_1 \to T_2}\phi$, $t' \in RED_{T_1 \to T_2}\phi$.

CR 3 Assume t is closed, $t \rightsquigarrow t'$ and $t' \in RED_{T_1 \to T_2}\phi$. Take arbitrary $u \in RED_{T_1}\phi$. By definition, we know $(t'\ u) \in RED_{T_2}\phi$. With our reduction strategy, $(t\ u) \rightsquigarrow (t'\ u)$. By IH(CR 1), u is closed, thus we know $(t\ u)$ is closed. By IH(CR 3), $(t\ u) \in RED_{T_2}\phi$. So by definition of $RED_{T_1 \to T_2}\phi$, $t \in RED_{T_1 \to T_2}\phi$.

CR 4 To show $I \subseteq RED_{T_1 \to T_2}\phi$, we need to show for arbitrary $i \in I$,$i \in RED_{T_1 \to T_2}\phi$. By definition of inactive terms, i is already in normal form. Take arbitrary $u \in RED_{T_1}\phi$. By IH(CR 1), $u \in N$ and closed. So $(i\ u) \overset{*}{\rightsquigarrow} (i\ u')$, where u' is the normal form of u. Thus by definition of inactive terms and IH(CR 4), $(i\ u') \in I \subseteq RED_{T_2}\phi$. $(i\ u)$ is closed, so by IH(CR 3), $(i\ u) \in RED_{T_2}\phi$. So by definition of $RED_{T_1 \to T_2}\phi$, $i \in RED_{T_1 \to T_2}\phi$. So $I \subseteq RED_{T_1 \to T_2}\phi$.

Step Case: $T = \Pi X.T$

CR 1 Assume $t \in RED_{\Pi X.T}$. By the fact in section 2.1, $\Re$ is non-empty. Take a arbitrary reducibility candidate R. By definition, $t \in RED_T\phi[R/X]$. By IH(CR 1), $t \in N$ and closed.

CR 2 Assume $t \in RED_{\Pi X.T}\phi$ and $t \rightsquigarrow t'$. Consider arbitrary reducibility candidate R. By definition, $t \in RED_T\phi[R/X]$. By IH(CR 2), $t' \in RED_T\phi[R/X]$. So by definition of $RED_{\Pi X.T}\phi[R/X]$, $t' \in RED_{\Pi X.T}\phi$.

CR 3 Assume t is closed, $t \rightsquigarrow t'$ and $t' \in RED_{\Pi X.T}\phi$. Take arbitrary reducibility candidate R. By definition, $t' \in RED_T\phi[R/X]$. We know that $t \rightsquigarrow t'$ and t is closed. So by IH(CR 3), $t \in RED_T\phi[R/X]$. So by definition, $t \in RED_{\Pi X.T}\phi$.

CR 4 We need to show that for arbitrary $i \in I$, $i \in RED_{\Pi X.T}\phi$. By definition, we actually need to show for arbitrary $R \in \Re$, $i \in RED_T\phi[R/X]$. By IH(CR 4), $I \subseteq RED_T\phi[R/X]$. So $i \in RED_T\phi[R/X]$. So it's the case.

3 Substitution Lemma

Substitution Lemma $RED_{[V/X]T}\phi = RED_T\phi[RED_V\phi/X]$.

Proof By induction on the structure of T.

Base Case: If $T = X$. We need to show $RED_V\phi = RED_X\phi[RED_V\phi/X]$. By definition, $RED_X\phi[RED_V\phi/X] = \phi[RED_V\phi/X](X) = RED_V\phi$. So it is the case.

Step Case: If $T = \Pi Y.W$. Then we need to show $RED_{(\Pi Y.[V/X]W)}\phi = RED_{\Pi Y.W}\phi[RED_V\phi/X]$. Take arbitrary $R \in \Re$ and arbitrary $t \in RED_{(\Pi Y.[V/X]W)}\phi$. By definition, $t \in RED_{[V/X]W}\phi[R/Y]$. By IH, $RED_{[V/X]W}\phi[R/Y] = RED_W\phi[R/Y, RED_V\phi[R/Y]/X] = RED_W\phi[R/Y, RED_V\phi/X]$.

So $t \in RED_W\phi[R/Y, RED_V\phi/X]$. By definition, $t \in RED_{\Pi Y.W}\phi[RED_V\phi/X]$.

Now let's prove the other direction. Take arbitrary $t \in RED_{\Pi Y.W}\phi[RED_V\phi/X]$ and arbitrary $R \in \Re$. By definition, $t \in RED_W\phi[RED_V\phi/X, R/Y]$. By IH, $RED_W\phi[RED_V\phi/X, R/Y] = RED_{[V/X]W}\phi[R/Y]$. So $t \in RED_{[V/X]W}\phi[R/Y]$. By definition, $t \in RED_{\Pi Y.[V/X]W}\phi$. So it is the case.

Step Case: If $T = T_1 \to T_2$. Then we need to show $RED_{([V/X]T_1 \to [V/X]T_2)}\phi = RED_{(T_1 \to T_2)}\phi[RED_V\phi/X]$. Take arbitrary $u \in RED_{([V/X]T_1)}\phi$ and $t \in RED_{([V/X]T_1 \to [V/X]T_2)}\phi$. By definition, $(t\ u) \in RED_{([V/X]T_2)}\phi$. By IH, $RED_{([V/X]T_1)}\phi = RED_{T_1}\phi[RED_V\phi/X]$ and $RED_{([V/X]T_2)}\phi = RED_{T_2}\phi[RED_V\phi/X]$. So $t \in RED_{(T_1 \to T_2)}\phi[RED_V\phi/X]$. The other direction is similar.

4 Reducibility Sets and Type assignment

Definition We define the set $[\Gamma]$ of well-typed substitutions (σ, δ) as follows:

$$\frac{}{(\emptyset, \emptyset) \in [.]}$$

$$\frac{(\sigma, \delta) \in [\Gamma] \quad R \in \Re}{(\sigma, \delta \cup \{(X, R)\}) \in [\Gamma, X : type]}$$

$$\frac{(\sigma, \delta) \in [\Gamma] \quad FV(T) \subseteq dom(\Gamma) \quad t \in RED_T\delta}{(\sigma \cup \{(x, t)\}, \delta) \in [\Gamma, x : T]}$$

Theorem If $\Gamma \vdash t : T$, then $\forall (\sigma, \delta) \in [\Gamma], (\sigma\ t) \in RED_T\delta$.

Proof By induction on the typing derivation of $\Gamma \vdash t : T$.

Base Case: The typing derivation looks like:

$$\frac{\Gamma(x) = T}{\Gamma \vdash x : T}$$

Because $\Gamma(x) = T$ and context is well-formed, $FV(T) \subseteq dom(\Gamma)$. By definition of $(\sigma, \delta) \in [\Gamma]$, we have $(x, t) \in \sigma$, where $t \in RED_T\delta$. So $(\sigma\ x) = t \in RED_T\delta$.

Application Case The typing derivation looks like:

$$\frac{\Gamma \vdash t_1 : (T_2 \to T_1) \quad \Gamma \vdash t_2 : T_2}{\Gamma \vdash t_1\ t_2 : T_1}$$

We need to prove that $\sigma(t_1\ t_2) \in RED_{T_1}\delta$. By IH, for any $(\sigma, \delta) \in [\Gamma], (\sigma\ t_1) \in\ RED_{(T_2 \to T_1)}\delta\ and\ (\sigma\ t_2) \in RED_{T_2}\delta$. By the defintion of $RED_{(T_2 \to T_1)}\delta$, we have $((\sigma t_1)(\sigma t_2)) = \sigma(t_1\ t_2) \in RED_{T_1}\delta$.

Lambda abstract Case The typing derivation looks like:

$$\frac{\Gamma, x : T_1\ \vdash t : T_2}{\Gamma \vdash \lambda x.t : (T_1 \to T_2)}$$

We need to show any $(\sigma, \delta) \in [\Gamma]$, we have $\sigma(\lambda x.t) = \lambda x.(\sigma\ t) \in RED_{(T_1 \to T_2)}\delta$. Since $\lambda x.(\sigma\ t) \in N$ and closed. By definition of $RED_{(T_1 \to T_2)}\delta$,we still need to show for arbitrary $u \in RED_{T_1}\delta, ((\lambda x.(\sigma\ t))\ u) \in RED_{T_2}\delta$. Since u is closed by CR 1, the normal form of u must be a value, which means $u \overset{*}{\rightsquigarrow} v$. So we have $(\lambda x.(\sigma\ t))\ u \overset{*}{\rightsquigarrow} (\lambda x.(\sigma\ t))\ v$, and by CR 2, $v \in RED_{T_1}\delta$. By definition of call-by-value reduction, $(\lambda x.(\sigma\ t))\ v \rightsquigarrow \sigma[v/x]t$. Since $v \in RED_{T_1}\delta$, and $FV(T_1) \subseteq dom(\Gamma)$, we have $(\sigma \cup \{(x, v)\}, \delta) \in [\Gamma, x : T_1]$. By IH, $(\sigma[v/x]t) \in RED_{T_2}\delta$. Since $((\lambda x.(\sigma\ t))\ u)$ is closed ,by CR 3, $((\lambda x.(\sigma\ t))\ u) \in RED_{T_2}\delta$. So by definition of $RED_{(T_1 \to T_2)}\delta$, $\sigma(\lambda x.t) = \lambda x.(\sigma\ t) \in RED_{(T_1 \to T_2)}\delta$.

Unviersal abstract Case The typing derivation looks like:

$$\frac{\Gamma, X : type\ \vdash t : T}{\Gamma \vdash t : \Pi X.T}$$

We need to show $\sigma(t) \in RED_{\Pi X.T}\delta$. By definition of $RED_{\Pi X.T}\delta$, we just need to show for arbitrary $R \in \Re$, $\sigma(t) \in RED_T\delta[R/X]$. By IH, for any $(\sigma, \delta \cup \{(X, R)\}) \in [\Gamma, X : type]$, $\sigma(t) \in RED_T\delta[R/X]$. So it is the case.

Unviersal application Case The typing derivation looks like:

$$\frac{\Gamma \vdash t : \Pi X.T}{\Gamma \vdash t : ([U/X]T)}$$

We need to show $\sigma(t) \in RED_{([U/X]T)}\delta$.By substitution lemma, $RED_{([U/X]T)}\delta = RED_T\delta[RED_U\delta/X]$. By IH, we know that $\sigma(t) \in RED_{(\Pi X.T)}\delta$. By definition, for arbitrary $R \in \Re, \sigma(t) \in RED_T\delta[R/X]$. We let $R = RED_U\delta$. So it is the case.

5 Conclusion

So for any closed term t, if $\Gamma \vdash t : T$, where $dom(\Gamma)$ only contains type variables, then $t \in RED_T\delta$, and by CR 1,$t \in N$.