Type Preservation for System F a la Curry

No Institute Given

1 System F

1.1 Syntax

Types $T \ ::= \ B \mid X \mid T_1 \rightarrow T_2 \mid \forall X.T$

Terms $t \ ::= c \mid x \mid (t_1 \ t_2) \mid \lambda x.t$

Reduction rules Contexts

$$C \ ::= * \mid C \ t \mid \lambda x.C \mid t \ C$$

Values $v \ ::= \lambda x.t \mid c$

Reductions

Full Beta Reduction.

$$C[(\lambda x.t) \ t'] \rightsquigarrow C[[t'/x]t]$$

1.2 Typing

Context $\Gamma ::= \cdot \mid \Gamma, x : T$

$$\frac{(x : T) \in \Gamma}{\Gamma \vdash x : T} \ T\_Var$$

$$\frac{\Gamma, x : T_1 \ \vdash t : T_2}{\Gamma \vdash \lambda x.t : T_1 \rightarrow T_2} \ \rightarrow\_intro$$

$$\frac{\Gamma \vdash t_1 : T_1 \rightarrow T_2 \quad \Gamma \vdash t_2 : T_1}{\Gamma \vdash t_1 \ t_2 : T_2} \ \rightarrow\_elim$$

$$\frac{\Gamma \vdash t : T \quad X \notin FV(\Gamma)}{\Gamma \vdash t : \forall X.T} \ \forall\_intro$$

Notice: $FV(\Gamma)$ is the set of all free type variables in Γ.

$$\frac{\Gamma \vdash t : \forall X.T}{\Gamma \vdash t : [T'/X]T} \ \forall\_elim$$

1.3 Basic Properties

Lemma 1.

- *If* $\Gamma \subseteq \Gamma'$ *and* $\Gamma \vdash t : T$*, then* $\Gamma' \vdash t : T$.
- *If* $\Gamma \vdash t : T$*, then* $fv(t) \subseteq dom(\Gamma)$*, where* $fv(t)$ *is the set of free term variable of* t.

This lemma should be straightforward by induction on derivation of $\Gamma \vdash t : T$.

Lemma 2. *If* $\Gamma \vdash t : T$*, then* $[T'/X]\Gamma \vdash t : [T'/X]T$.

Proof. By induction on the derivation of $\Gamma \vdash t : T$.

Base Case:

$$\frac{(x : T) \in \Gamma}{\Gamma \vdash x : T} \; T\_Var$$

Obvious.

Step Case:

$$\frac{\Gamma, x : T_1 \;\vdash t : T_2}{\Gamma \vdash \lambda x.t : T_1 \to T_2} \to\_intro$$

We want to show $[T'/X]\Gamma \vdash \lambda x.t : [T'/X]T_1 \to [T'/X]T_2$. By IH, we have $[T'/X]\Gamma, x : [T'/X]T_1 \;\vdash t : [T'/X]T_2$. Thus it is the case.

Step Case:

$$\frac{\Gamma \vdash t_1 : T_1 \to T_2 \qquad \Gamma \vdash t_2 : T_1}{\Gamma \vdash t_1\ t_2 : T_2} \to\_elim$$

Again, by IH, we have $[T'/X]\Gamma \vdash t_1 : [T'/X]T_1 \to [T'/X]T_2$ and $[T'/X]\Gamma \vdash t_2 : [T'/X]T_1$. Thus it is the case.

Step Case:

$$\frac{\Gamma \vdash t : T \quad X \notin FV(\Gamma)}{\Gamma \vdash t : \forall X.T} \; \forall\_intro$$

We want to show $[T'/Y]\Gamma \vdash t : \forall X.[T'/Y]T$. By IH, we have $[T'/Y]\Gamma \vdash t : [T'/Y]T$. Of course, $X \notin FV(T')$. So it is the case.

Step Case:

$$\frac{\Gamma \vdash t : \forall X.T}{\Gamma \vdash t : [T'/X]T} \; \forall\_elim$$

We want to show that $[T''/Y]\Gamma \vdash t : [T''/Y]([T'/X]T)$. By IH, we have $[T''/Y]\Gamma \vdash t : \forall X.[T''/Y]T$. By proper renaming variable, it is also the case.

Lemma 3. *If* $\Gamma, x : T_1 \vdash t : T_2$ *and* $\Gamma \vdash t' : T_1$*, then* $\Gamma \vdash [t'/x]t : T_2$.

Proof. By induction on the derivation of $\Gamma, x : T_1 \vdash t : T_2$.

Base Case:

$$\frac{(x : T) \in \Gamma}{\Gamma, x : T \vdash x : T} \; T\_Var$$

Obvious.

Step Case:

$$\frac{\Gamma, y : T, x : T_1 \ \vdash t : T_2}{\Gamma, y : T \vdash \lambda x.t : T_1 \to T_2} \; \to\_intro$$

We know that $\Gamma \vdash t' : T$. We want to show $\Gamma \vdash \lambda x.[t'/y]t : T_1 \to T_2$. By IH, we have $\Gamma, x : T_1 \ \vdash [t'/y]t : T_2$. Thus it is the case.

Step Case:

$$\frac{\Gamma, x : T \vdash t_1 : T_1 \to T_2 \quad \Gamma, x : T \vdash t_2 : T_1}{\Gamma, x : T \vdash t_1 \ t_2 : T_2} \; \to\_elim$$

We have $\Gamma \vdash t' : T$. By IH, we have $\Gamma \vdash [t'/x]t_1 : T_1 \to T_2$ and $\Gamma \vdash [t'/x]t_2 : T_1$. Thus we have $\Gamma \vdash [t'/x](t_1 \ t_2) : T_2$. So it is the case.

Step Case:

$$\frac{\Gamma, x : T' \vdash t : T \quad X \notin FV(\Gamma)}{\Gamma, x : T' \vdash t : \forall X.T} \; \forall\_intro$$

We know that $\Gamma \vdash t' : T'$. By IH, we have $\Gamma \vdash [t'/x]t : T$. Thus we have $\Gamma \vdash [t'/x]t : \forall X.T$.

Step Case:

$$\frac{\Gamma, x : T'' \vdash t : \forall X.T}{\Gamma, x : T'' \vdash t : [T'/X]T} \; \forall\_elim$$

We know that $\Gamma \vdash t' : T''$. By IH, we have $\Gamma \vdash [t'/x]t : \forall X.T$. Thus we have $\Gamma \vdash [t'/x]t : [T'/X]T$.

1.4 Barendregt's Order

Definition 1. *$\Gamma \vdash T_1 > T_2$ iff $\exists t, \Gamma \vdash t : T_1$ and*

$T_1 \equiv \forall X.T$ and $T_2 \equiv [T'/X]T$. (By proper renaming, we have $FV(T') \cap FV(\Gamma) = \emptyset$.) Or

$T_2 \equiv \forall X.T_1$, where $X \notin FV(\Gamma)$.

Notice: we write $\geq$ as the transitive and reflexive closure of $>$.

Lemma 4. *If $\Gamma \vdash T \geq T'$ and $\Gamma \vdash t : T$, then $\Gamma \vdash t : T'$.*

Proof. Since $\Gamma \vdash T \geq T'$ implies $\Gamma \vdash T_1 \equiv T_1 > T_2 > ... > T_n \equiv T'$. According to the definition of $\Gamma \vdash T_i > T_{i+1}$, if $\Gamma \vdash t : T_i$, then $\Gamma \vdash t : T_{i+1}$. Thus we finally can get $\Gamma \vdash t : T'$.

Lemma 5.

1. *If* $\Gamma \vdash x : T$*, then* $\exists T', \Gamma \vdash T' \geq T$ *and* $(x : T') \in \Gamma$.
2. *If* $\Gamma \vdash t_1 t_2 : T$*, then* $\exists T', \Gamma \vdash T' \geq T$*, and* $\exists T'', \Gamma \vdash t_2 : T'', \Gamma \vdash t_1 : T'' \to T'$.
3. *If* $\Gamma \vdash \lambda x.t : T$*, then* $\exists T', T'', \Gamma, x : T' \vdash t : T''$ *and* $\Gamma \vdash T' \to T'' \geq T$.

Proof. By induction on derivation.

1. Base Case:

$$\frac{(x : T) \in \Gamma}{\Gamma \vdash x : T}\ T\_Var$$

By reflexity, we know that it is true.

Step Case:

$$\frac{\Gamma \vdash x : T' \quad X \notin FV(\Gamma)}{\Gamma \vdash x : T}\ \forall\_intro$$

where $T \equiv \forall X.T'$. Thus by definition, we have $\Gamma \vdash T' > T$. By IH, we have $\exists T'', \Gamma \vdash T'' \geq T', (x : T'') \in \Gamma$. By transitivity, we have $\Gamma \vdash T'' \geq T$. Thus it is the case.

Step Case:

$$\frac{\Gamma \vdash x : \forall X.T'}{\Gamma \vdash x : T}\ \forall\_elim$$

where $T \equiv [T''/X]T'$. By definition, we have $\Gamma \vdash \forall X.T' > T$. By IH, $\exists T_1, \Gamma \vdash T_1 \geq \forall X.T'$ and $(x : T_1) \in \Gamma$. By transitivity, we have $\Gamma \vdash T_1 \geq T$. Thus it is the case.

2. Base Case:

$$\frac{\Gamma \vdash t_1 : T_1 \to T_2 \quad \Gamma \vdash t_2 : T_1}{\Gamma \vdash t_1\ t_2 : T_2}\ \to\_elim$$

It is trivial true by reflexity.

Step Case:

$$\frac{\Gamma \vdash t_1 t_2 : T' \quad X \notin FV(\Gamma)}{\Gamma \vdash t_1 t_2 : T}\ \forall\_intro$$

$T \equiv \forall X.T'$. By definition, $\Gamma \vdash T' > T$. By IH, $\exists T_1, \exists T_2, \Gamma \vdash T_2 \geq T', \Gamma \vdash t_1 : T_1 \to T_2$ and $\Gamma \vdash t_2 : T_1$. By transitivity, we have $\Gamma \vdash T_2 \geq T$. Thus it is the case.

Step Case:

$$\frac{\Gamma \vdash t_1 t_2 : \forall X.T''}{\Gamma \vdash t_1 t_2 : [T'/X]T''}\ \forall\_elim$$

$T \equiv [T'/X]T''$. By definition, $\Gamma \vdash \forall X.T'' > T$. By IH, $\exists T_1, \exists T_2, \Gamma \vdash T_2 \geq \forall X.T'', \Gamma \vdash t_1 : T_1 \to T_2$ and $\Gamma \vdash t_2 : T_1$. By transitivity, we have $\Gamma \vdash T_2 \geq T$. Thus it is the case.

3. Base Case:

$$\frac{\Gamma, x : T_1 \ \vdash t : T_2}{\Gamma \vdash \lambda x.t : T_1 \to T_2} \to\_intro$$

It is true by reflexity.

Step Case:

$$\frac{\Gamma \vdash \lambda x.t : T' \quad X \notin FV(\Gamma)}{\Gamma \vdash \lambda x.t : T} \forall\_intro$$

$T \equiv \forall X.T'$. By definition, $\Gamma \vdash T' > T$. By IH, $\exists T_1, \exists T_2, \Gamma \vdash T_1 \to T_2 \geq T', \Gamma, x : T_1 \vdash t : T_2$. By transitivity, we have $\Gamma \vdash T_1 \to T_2 \geq T$. Thus it is the case.

Step Case:

$$\frac{\Gamma \vdash \lambda x.t : \forall X.T''}{\Gamma \vdash \lambda x.t : [T'/X]T''} \forall\_elim$$

$T \equiv [T'/X]T''$. By definition, $\Gamma \vdash \forall X.T'' > T$. By IH, $\exists T_1, \exists T_2, \Gamma \vdash T_1 \to T_2 \geq \forall X.T'', \Gamma, x : T_1 \vdash t : T_2$. By transitivity, we have $\Gamma \vdash T_1 \to T_2 \geq T$. Thus it is the case.

Definition 2. *We define o is a function from types to types.*

$o(X) := X$.

$o(T_1 \to T_2) := T_1 \to T_2$.

$o(\forall X.T) := o(T)$.

Lemma 6.

1. *For any T, T_1, $\exists T_2, o([T_1/X]T) \equiv [T_2/X]o(T)$.*
2. *If $\Gamma \vdash T_1 \geq T_2$, then $\exists \sigma, o(T_2) \equiv \sigma(o(T_1))$ and $dom(\sigma) \cap FV(\Gamma) = \emptyset$.*
3. *If $\Gamma \vdash T_1 \to T_2 \geq T_1' \to T_2'$, then $\exists \sigma, T_1' \to T_2' \equiv \sigma(T_1 \to T_2)$ and $dom(\sigma) \cap FV(\Gamma) = \emptyset$.*

Notice that σ here is a type substitution.

Proof. 1. By induction on the structure of T.

Base Case: $T \equiv X$, then $o([T_1/X]X) \equiv o(T_1)$. Thus $T_2 \equiv o(T_1)$ here.

Step Case: $T \equiv T_a \to T_b$. We have $o([T_1/X]T) \equiv o([T_1/X]T_a \to [T_1/X]T_b) \equiv [T_1/X]T_a \to [T_1/X]T_b \equiv [T_1/X]o(T_a \to T_b)$. So here $T_2 \equiv T_1$.

Step Case: $T \equiv \forall Y.T'$. We have $o([T_1/X]\forall Y.T') \equiv o([T_1/X]T')$. Then by IH, $\exists T'', o([T_1/X]T') \equiv [T''/X]o(T') \equiv [T''/X]o(\forall Y.T')$. Thus $T_2 \equiv T''$ here.

2. We will prove this by induction on the length of $\Gamma \vdash T_1 \geq T_2$.

Base Case: $\Gamma \vdash T_1 > T_2$. We want to show $\exists \sigma, o(T_2) \equiv \sigma(o(T_1))$ and $dom(\sigma) \cap FV(\Gamma) = \emptyset$. $\Gamma \vdash T_1 > T_2$ implies either $T_2 \equiv \forall Y.T_1$ or $T_1 \equiv \forall Y.T', T_2 \equiv [T''/Y]T'$. If $T_2 \equiv \forall Y.T_1$, then $o(T_2) \equiv o(T_1)$. Thus in this case $\sigma = \emptyset$. If $T_1 \equiv \forall Y.T', T_2 \equiv [T''/Y]T'$, then $o(T_2) \equiv o([T''/Y]T')$. By 1, we have $\exists T_x, o([T''/Y]T') \equiv [T_x/Y]o(T') \equiv [T_x/Y]o(\forall Y.T') \equiv [T_x/Y]o(T_1)$. Thus $\sigma = [T_x/Y]$. By definition of $\Gamma \vdash T_1 > T_2$, we know that $Y \notin FV(\Gamma)$, thus $dom(\sigma) \cap FV(\Gamma) = \emptyset$.

Step Case: $\Gamma \vdash T_1 > ... > T_u > T_2$. By IH, we have $\exists \sigma, o(T_u) \equiv \sigma(o(T_1))$ and $dom(\sigma) \cap FV(\Gamma) = \emptyset$. If $T_2 \equiv \forall X.T_u$. Then $o(T_2) \equiv o(T_u)$, σ is what we want here. If $T_u \equiv \forall X.T$ *and* $T_2 \equiv [T'/X]T$, then $o(T_2) \equiv o([T'/X]T)$. By 1, we have $\exists T'', o([T'/X]T) \equiv [T''/X]o(T) \equiv [T''/X]o(\forall X.T) \equiv [T''/X]o(T_u) \equiv [T''/X](\sigma(o(T_1)))$. Thus $[T''/X](\sigma)$ is what we want. Since $X \notin FV(\Gamma), dom(\sigma) \cap FV(\Gamma) = \emptyset$, we have $dom([T''/X](\sigma)) \cap FV(\Gamma) = \emptyset$.

3. We have $\Gamma \vdash T_1 \to T_2 \geq T_1' \to T_2'$ and $T_1' \to T_2' \equiv o(T_1' \to T_2')$. By 2, we have $\exists \sigma, o(T_1' \to T_2') \equiv \sigma(o(T_1 \to T_2)) \equiv \sigma(T_1 \to T_2)$ and $dom(\sigma) \cap FV(\Gamma) = \emptyset$. So $T_1' \to T_2' \equiv \sigma(T_1 \to T_2)$.

Theorem 1. *If $t \rightsquigarrow t'$ and $\Gamma \vdash t : T$, then $\Gamma \vdash t' : T$.*

Proof. By induction on the derivation of $t \rightsquigarrow t'$. We will just prove when $t \equiv (\lambda x.t')t''$. So we have $\Gamma \vdash (\lambda x.t')t'' : T$. By lemma 5.2, we have $\exists T', T'', \Gamma \vdash T' \geq T, \Gamma \vdash \lambda x.t' : T'' \to T'$ and $\Gamma \vdash t'' : T''$. By lemma 5.3, we have $\exists S_1, S_2, \Gamma \vdash S_1 \to S_2 \geq T'' \to T'$ and $\Gamma, x : S_1 \vdash t' : S_2$. By lemma 6.3, we have $\exists \sigma, \sigma(S_1 \to S_2) \equiv T'' \to T'$ and $dom(\sigma) \cap FV(\Gamma) = \emptyset$. So by lemma 2, we have $\Gamma, x : T'' \vdash t' : T'$. By lemma 3, we have $\Gamma \vdash [t''/x]t' : T'$. Since $\Gamma \vdash T' \geq T$, by lemma 4, we have $\Gamma \vdash [t''/x]t' : T$. Thus we get type preservation.